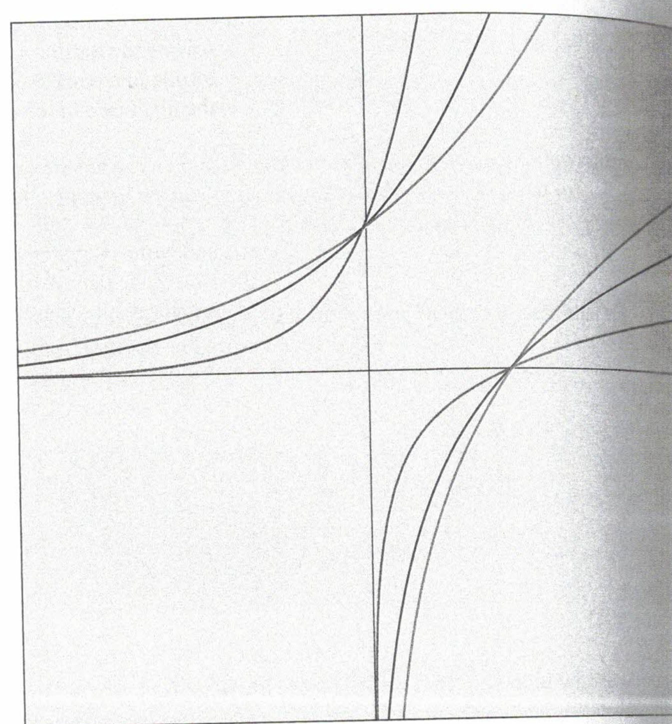
7

INVERSE FUNCTIONS:

EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS



The exponential and logarithmic functions are inverse functions of each other.

The common theme that links the functions of this chapter is that they occur as pairs of inverse functions. In particular, two of the most important functions that occur in mathematics and its applications are the exponential function $f(x) = a^x$ and its inverse function, the logarithmic function $g(x) = \log_a x$. In this chapter we investigate their properties, compute their derivatives, and use them to describe exponential growth and decay in biology, physics, chemistry, and other sciences. We also study the inverses of trigonometric and hyperbolic functions. Finally, we look at a method (l'Hospital's Rule) for computing difficult limits and apply it to sketching curves.

There are two possible ways of defining the exponential and logarithmic functions and developing their properties and derivatives. One is to start with the exponential function (defined as in algebra or precalculus courses) and then define the logarithm as its inverse. That is the approach taken in Sections 7.2, 7.3, and 7.4 and is probably the most intuitive method. The other way is to start by defining the logarithm as an integral and then define the exponential function as its inverse. This approach is followed in Sections 7.2*, 7.3*, and 7.4* and, although it is less intuitive, many instructors prefer it because it is more rigorous and the properties follow more easily. You need only read one of these two approaches (whichever your instructor recommends).

7.1 INVERSE FUNCTIONS

Table 1 gives data from an experiment in which a bacteria culture started with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t: $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the *inverse function* of f, denoted by f^{-1}, and read "f inverse." Thus $t = f^{-1}(N)$ is the time required for the population level to reach N. The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

TABLE 1 N as a function of t

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

TABLE 2 t as a function of N

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

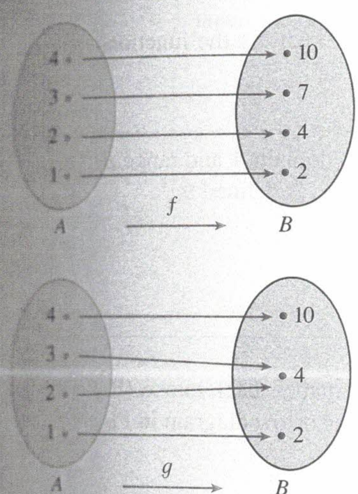


FIGURE 1
f is one-to-one; g is not

Not all functions possess inverses. Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4). In symbols,

$$g(2) = g(3)$$

but

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Functions that share this property with f are called *one-to-one functions.*

▪ In the language of inputs and outputs, this definition says that f is one-to-one if each output corresponds to only one input.

1 DEFINITION A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

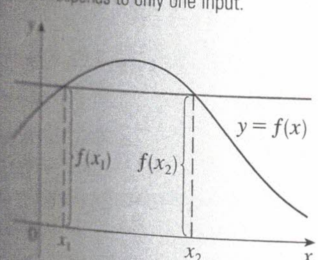


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

HORIZONTAL LINE TEST A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLE 1 Is the function $f(x) = x^3$ one-to-one?

SOLUTION 1 If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by Definition 1, $f(x) = x^3$ is one-to-one.

SOLUTION 2 From Figure 3 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one. □

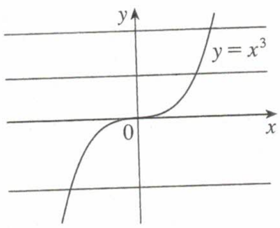


FIGURE 3
$f(x) = x^3$ is one-to-one.

EXAMPLE 2 Is the function $g(x) = x^2$ one-to-one?

SOLUTION 1 This function is not one-to-one because, for instance,

$$g(1) = 1 = g(-1)$$

and so 1 and -1 have the same output.

SOLUTION 2 From Figure 4 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one. □

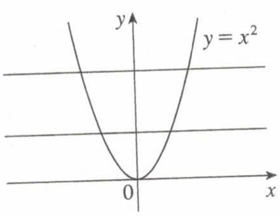


FIGURE 4
$g(x) = x^2$ is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

> **2 DEFINITION** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \iff f(x) = y$$
>
> for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.) The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f. Note that

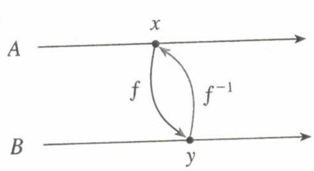


FIGURE 5

> domain of f^{-1} = range of f
>
> range of f^{-1} = domain of f

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

The reciprocal $1/f(x)$ could, however, be written as $[f(x)]^{-1}$.

EXAMPLE 3 If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$
$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$
$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

The diagram in Figure 6 makes it clear how f^{-1} reverses the effect of f in this case.

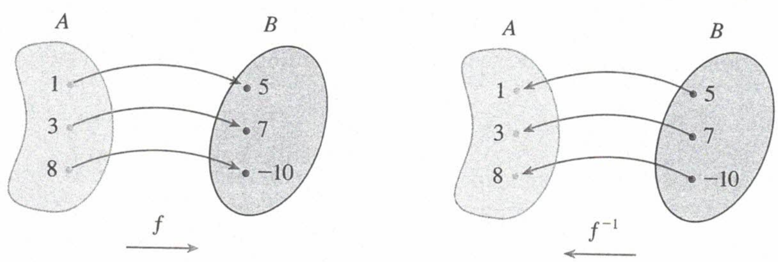


FIGURE 6
The inverse function reverses inputs and outputs.

□

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

3
$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

4
$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7). Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

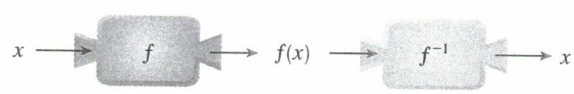


FIGURE 7

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$
$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$. If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

> **5 HOW TO FIND THE INVERSE FUNCTION OF A ONE-TO-ONE FUNCTION f**
>
> STEP 1 Write $y = f(x)$.
>
> STEP 2 Solve this equation for x in terms of y (if possible).
>
> STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

V EXAMPLE 4 Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

■ In Example 4, notice how f^{-1} reverses the effect of f. The function f is the rule "Cube, then add 2"; f^{-1} is the rule "Subtract 2, then take the cube root."

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

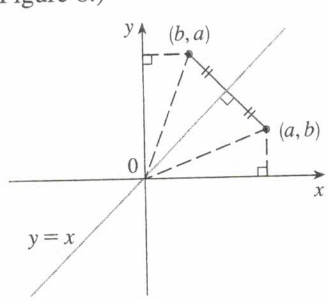


FIGURE 8

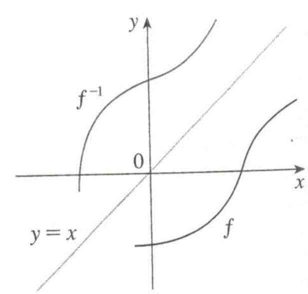


FIGURE 9

Therefore, as illustrated by Figure 9:

> The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

EXAMPLE 5 Sketch the graphs of $f(x) = \sqrt{-1 - x}$ and its inverse function using the same coordinate axes.

SOLUTION First we sketch the curve $y = \sqrt{-1 - x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}. (See Figure 10.) As a check on our graph, notice that the expression for f^{-1} is $f^{-1}(x) = -x^2 - 1$, $x \geqslant 0$. So the graph of f^{-1} is the right half of the parabola $y = -x^2 - 1$ and this seems reasonable from Figure 10.

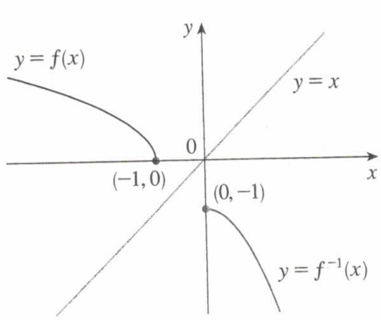


FIGURE 10

THE CALCULUS OF INVERSE FUNCTIONS

Now let's look at inverse functions from the point of view of calculus. Suppose that f is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.) Since the graph of f^{-1} is obtained from the graph of f by reflecting about the line $y = x$, the graph of f^{-1} has no break in it either (see Figure 9). Thus we might expect that f^{-1} is also a continuous function.

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible. A proof can be found in Appendix F.

> **6 THEOREM** If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

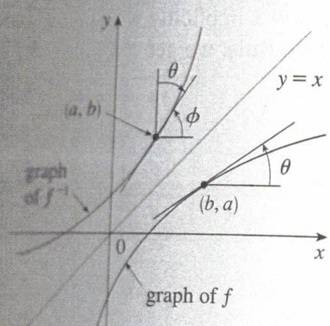


FIGURE 11

Now suppose that f is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it. We get the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or kink in it either. We therefore expect that f^{-1} is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of f^{-1} at a given point by a geometric argument. In Figure 11 the graphs of f and its inverse f^{-1} are shown. If $f(b) = a$, then $f^{-1}(a) = b$ and $(f^{-1})'(a)$ is the slope of the tangent to the graph of f^{-1} at (a, b), which is $\tan \phi$. Likewise, $f'(b) = \tan \theta$. From Figure 11 we see that $\theta + \phi = \pi/2$, so

$$(f^{-1})'(a) = \tan \phi = \tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta = \frac{1}{\tan \theta} = \frac{1}{f'(b)}$$

that is,

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

> **7 THEOREM** If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and
>
> $$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

PROOF Write the definition of derivative as in Equation 3.1.5:

$$(f^{-1})'(a) = \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$$

If $f(b) = a$, then $f^{-1}(a) = b$. And if we let $y = f^{-1}(x)$, then $f(y) = x$. Since f is differentiable, it is continuous, so f^{-1} is continuous by Theorem 6. Thus if $x \to a$, then $f^{-1}(x) \to f^{-1}(a)$, that is, $y \to b$. Therefore

$$\begin{aligned}(f^{-1})'(a) &= \lim_{x \to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\dfrac{f(y) - f(b)}{y - b}} = \frac{1}{\displaystyle\lim_{y \to b} \frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}\end{aligned}$$

☐

Note that $x \neq a \Rightarrow f(y) \neq f(b)$ because f is one-to-one.

NOTE 1 Replacing a by the general number x in the formula of Theorem 7, we get

8 $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

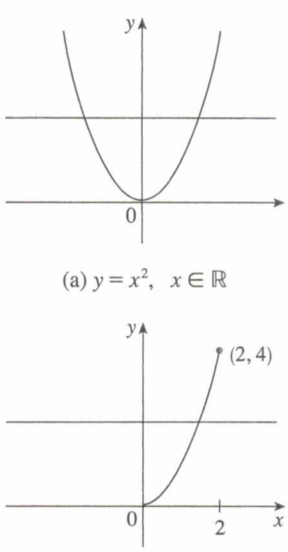
If we write $y = f^{-1}(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

NOTE 2 If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, remembering that y is a function of x, and using the Chain Rule, we get

$$f'(y)\frac{dy}{dx} = 1$$

Therefore $$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\frac{dx}{dy}}$$



(a) $y = x^2$, $x \in \mathbb{R}$

(b) $f(x) = x^2$, $0 \leq x \leq 2$

FIGURE 12

EXAMPLE 6 Although the function $y = x^2$, $x \in \mathbb{R}$, is not one-to-one and therefore does not have an inverse function, we can turn it into a one-to-one function by restricting its domain. For instance, the function $f(x) = x^2$, $0 \leq x \leq 2$, is one-to-one (by the Horizontal Line Test) and has domain $[0, 2]$ and range $[0, 4]$. (See Figure 12.) Thus f has an inverse function f^{-1} with domain $[0, 4]$ and range $[0, 2]$.

Without computing a formula for $(f^{-1})'$ we can still calculate $(f^{-1})'(1)$. Since $f(1) = 1$, we have $f^{-1}(1) = 1$. Also $f'(x) = 2x$. So by Theorem 7 we have

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{2}$$

In this case it is easy to find f^{-1} explicitly. In fact, $f^{-1}(x) = \sqrt{x}$, $0 \leq x \leq 4$. [In general, we could use the method given by (5).] Then $(f^{-1})'(x) = 1/(2\sqrt{x})$, so $(f^{-1})'(1) = \frac{1}{2}$, which agrees with the preceding computation. The functions f and f^{-1} are graphed in Figure 13. □

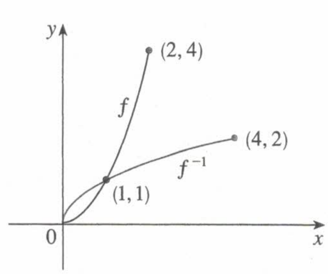


FIGURE 13

V EXAMPLE 7 If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

SOLUTION Notice that f is one-to-one because

$$f'(x) = 2 - \sin x > 0$$

and so f is increasing. To use Theorem 7 we need to know $f^{-1}(1)$ and we can find it by inspection:

$$f(0) = 1 \Rightarrow f^{-1}(1) = 0$$

Therefore $$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$ □

7.1 EXERCISES

1. (a) What is a one-to-one function?
   (b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
   (b) If you are given a formula for f, how do you find a formula for f^{-1}?
   (c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

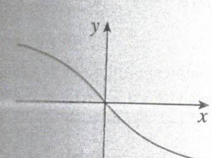
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

5.



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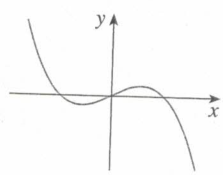

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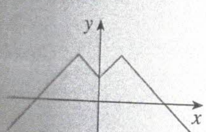

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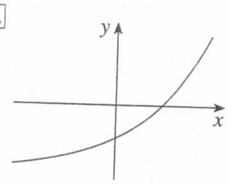


9. $f(x) = x^2 - 2x$

10. $f(x) = 10 - 3x$

11. $g(x) = 1/x$

12. $g(x) = |x|$

13. $h(x) = 1 + \cos x$

14. $h(x) = 1 + \cos x, \quad 0 \leqslant x \leqslant \pi$

15. $f(t)$ is the height of a football t seconds after kickoff.

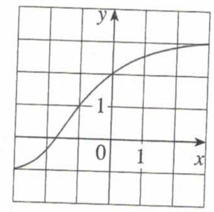
16. $f(t)$ is your height at age t.

17. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

18. If $f(x) = x + \cos x$, find $f^{-1}(1)$.

19. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

20. The graph of f is given.
   (a) Why is f one-to-one?
   (b) What are the domain and range of f^{-1}?
   (c) What is the value of $f^{-1}(2)$?
   (d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. $f(x) = 3 - 2x$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = \sqrt{10 - 3x}$

26. $y = 2x^3 + 3$

27. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

28. $f(x) = 2x^2 - 8x, \quad x \geqslant 2$

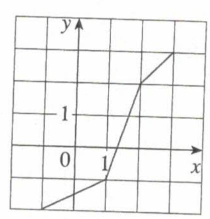
29–30 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about this line.

29. $f(x) = x^4 + 1, \quad x \geqslant 0$

30. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

31–32 Use the given graph of f to sketch the graph of f^{-1}.

31.



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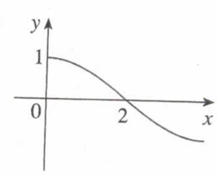


33–36
(a) Show that f is one-to-one.
(b) Use Theorem 7 to find $(f^{-1})'(a)$.
(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

(d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).
(e) Sketch the graphs of f and f^{-1} on the same axes.

33. $f(x) = x^3, \quad a = 8$

34. $f(x) = \sqrt{x-2}, \quad a = 2$

35. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

36. $f(x) = 1/(x-1), \quad x > 1, \quad a = 2$

37–40 Find $(f^{-1})'(a)$.

37. $f(x) = 2x^3 + 3x^2 + 7x + 4, \quad a = 4$

38. $f(x) = x^3 + 3\sin x + 2\cos x, \quad a = 2$

39. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

40. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \quad a = 2$

41. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

42. Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

CAS **43.** Use a computer algebra system to find an explicit expression for the inverse function $f(x) = \sqrt{x^3 + x^2 + x + 1}$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

44. Show that $h(x) = \sin x$, $x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one. Compute the derivative of $f^{-1} = \sin^{-1}$ by the method of Note 2.

45. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

46. (a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

(b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

7.2 EXPONENTIAL FUNCTIONS AND THEIR DERIVATIVES

If your instructor has assigned Sections 7.2*, 7.3*, and 7.4*, you don't need to read Sections 7.2–7.4 (pp. 392–421).

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

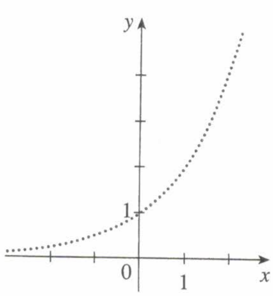


FIGURE 1
Representation of $y = 2^x$, x rational

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing continuous function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$\begin{aligned}
1.73 < \sqrt{3} < 1.74 \quad &\Rightarrow \quad 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 \quad &\Rightarrow \quad 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 \quad &\Rightarrow \quad 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 \quad &\Rightarrow \quad 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206} \\
\vdots \qquad\quad \vdots \qquad\quad &\qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

A proof of this fact is given in J. Marsden and A. Weinstein, *Calculus Unlimited* (Menlo Park, CA: Benjamin/Cummings, 1981). For an online version, see www.cds.caltech.edu/~marsden/volume/cu/CU.pdf

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad \ldots$$

and less than all of the numbers

$$2^{1.8}, \quad 2^{1.74}, \quad 2^{1.733}, \quad 2^{1.7321}, \quad 2^{1.73206}, \quad \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the preceding approximation process we can compute it correct to six decimal places:

$$2^{\sqrt{3}} \approx 3.321997$$

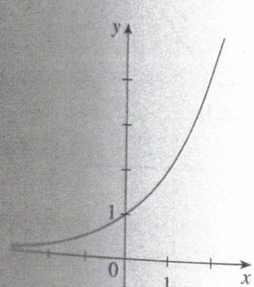


FIGURE 2
$y = 2^x$, x real

Similarly, we can define 2^x (or a^x, if $a > 0$) where x is any irrational number. Figure 2 shows how all the holes in Figure 1 have been filled to complete the graph of the function $f(x) = 2^x$, $x \in \mathbb{R}$.

In general, if a is any positive number, we define

1 $$a^x = \lim_{r \to x} a^r \qquad r \text{ rational}$$

This definition makes sense because any irrational number can be approximated as closely as we like by a rational number. For instance, because $\sqrt{3}$ has the decimal representation $\sqrt{3} = 1.7320508\ldots$, Definition 1 says that $2^{\sqrt{3}}$ is the limit of the sequence of numbers

$$2^{1.7}, \quad 2^{1.73}, \quad 2^{1.732}, \quad 2^{1.7320}, \quad 2^{1.73205}, \quad 2^{1.732050}, \quad 2^{1.7320508}, \quad \ldots$$

Similarly, 5^{π} is the limit of the sequence of numbers

$$5^{3.1}, \quad 5^{3.14}, \quad 5^{3.141}, \quad 5^{3.1415}, \quad 5^{3.14159}, \quad 5^{3.141592}, \quad 5^{3.1415926}, \quad \ldots$$

It can be shown that Definition 1 uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

The graphs of members of the family of functions $y = a^x$ are shown in Figure 3 for various values of the base a. Notice that all of these graphs pass through the same point because $a^0 = 1$ for $a \neq 0$. Notice also that as the base a gets larger, the exponential function grows more rapidly (for $x > 0$).

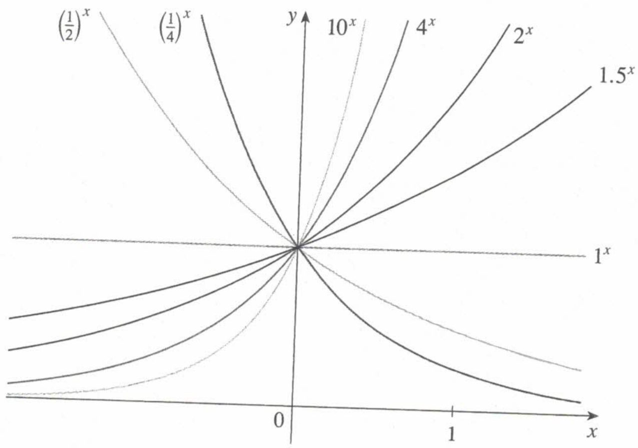


FIGURE 3
Members of the family of exponential functions

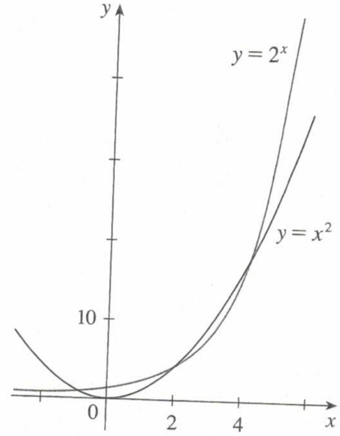


FIGURE 4

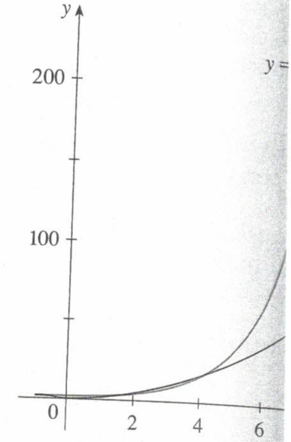
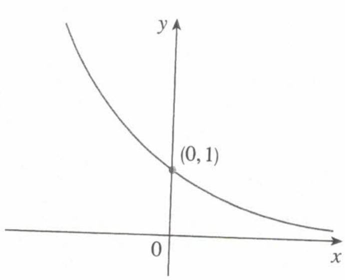


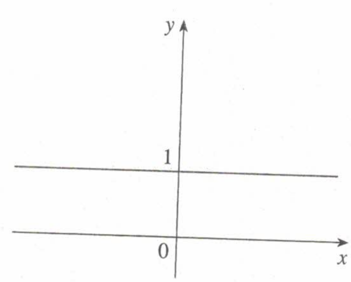
FIGURE 5

Figure 4 shows how the exponential function $y = 2^x$ compares with the power function $y = x^2$. The graphs intersect three times, but ultimately the exponential curve grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

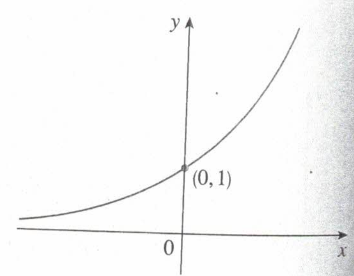
You can see from Figure 3 that there are basically three kinds of exponential functions $y = a^x$. If $0 < a < 1$, the exponential function decreases; if $a = 1$, it is a constant; and if $a > 1$, it increases. These three cases are illustrated in Figure 6. Since $(1/a)^x = 1/a^x = a^{-x}$, the graph of $y = (1/a)^x$ is just the reflection of the graph of $y = a^x$ about the y-axis.



(a) $y = a^x$, $0 < a < 1$



(b) $y = 1^x$



(c) $y = a^x$, $a > 1$

FIGURE 6

The properties of the exponential function are summarized in the following theorem.

> **2 THEOREM** If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $0 < a < 1$, $f(x) = a^x$ is a decreasing function; if $a > 1$, f is an increasing function. If $a, b > 0$ and $x, y \in \mathbb{R}$, then
>
> **1.** $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = \dfrac{a^x}{a^y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

www.stewartcalculus.com
For review and practice using the Laws of Exponents, click on *Review of Algebra*.

The reason for the importance of the exponential function lies in properties 1–4, which are called the **Laws of Exponents**. If x and y are rational numbers, then these laws are well known from elementary algebra. For arbitrary real numbers x and y these laws can be deduced from the special case where the exponents are rational by using Equation 1.

The following limits can be read from the graphs shown in Figure 6 or proved from the definition of a limit at infinity. (See Exercise 69 in Section 7.3.)

3 If $a > 1$, then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$

If $0 < a < 1$, then $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$

In particular, if $a \neq 1$, then the x-axis is a horizontal asymptote of the graph of the exponential function $y = a^x$.

EXAMPLE 1

(a) Find $\lim_{x\to\infty} (2^{-x} - 1)$.

(b) Sketch the graph of the function $y = 2^{-x} - 1$.

SOLUTION

(a)

$$\begin{aligned} \lim_{x\to\infty} (2^{-x} - 1) &= \lim_{x\to\infty} \left[\left(\tfrac{1}{2}\right)^x - 1\right] \\ &= 0 - 1 \qquad \left[\text{by (3) with } a = \tfrac{1}{2} < 1\right] \\ &= -1 \end{aligned}$$

(b) We write $y = \left(\frac{1}{2}\right)^x - 1$ as in part (a). The graph of $y = \left(\frac{1}{2}\right)^x$ is shown in Figure 3, so we shift it down one unit to obtain the graph of $y = \left(\frac{1}{2}\right)^x - 1$ shown in Figure 7. (For a review of shifting graphs, see Section 1.3.) Part (a) shows that the line $y = -1$ is a horizontal asymptote.

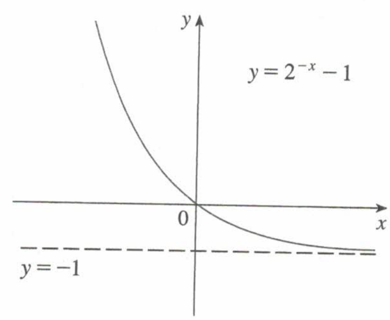


FIGURE 7

□

APPLICATIONS OF EXPONENTIAL FUNCTIONS

The exponential function occurs very frequently in mathematical models of nature and society. Here we indicate briefly how it arises in the description of population growth. In Section 7.5 we will pursue these and other applications in greater detail.

In Section 3.7 we considered a bacteria population that doubles every hour and saw that if the initial population is n_0, then the population after t hours is given by the function $f(t) = n_0 2^t$. This population function is a constant multiple of the exponential function

$y = 2^t$, so it exhibits the rapid growth that we observed in Figures 2 and 5. Under conditions (unlimited space and nutrition and freedom from disease), this expon growth is typical of what actually occurs in nature.

What about the human population? Table 1 shows data for the population of the in the 20th century and Figure 8 shows the corresponding scatter plot.

TABLE 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

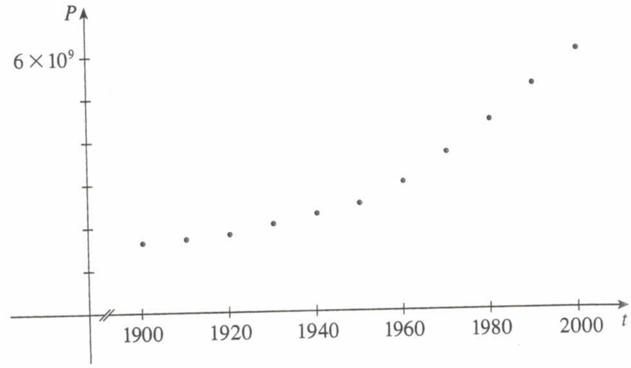


FIGURE 8 Scatter plot for world population growth

The pattern of the data points in Figure 8 suggests exponential growth, so w graphing calculator with exponential regression capability to apply the method squares and obtain the exponential model

$$P = (0.008079266) \cdot (1.013731)^t$$

Figure 9 shows the graph of this exponential function together with the origi points. We see that the exponential curve fits the data reasonably well. The perio tively slow population growth is explained by the two world wars and the Great sion of the 1930s.

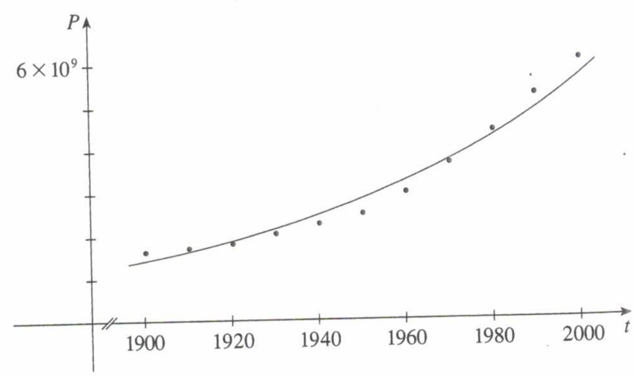


FIGURE 9 Exponential model for population growth

DERIVATIVES OF EXPONENTIAL FUNCTIONS

Let's try to compute the derivative of the exponential function $f(x) = a^x$ using tion of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

The factor a^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

4 $$f'(x) = f'(0)a^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself.* (The slope is proportional to the height.)

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $a = 2$ and $a = 3$. (Values are stated correct to four decimal places.) It appears that the limits exist and

$$\text{for } a = 2, \quad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$$

$$\text{for } a = 3, \quad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10$$

In fact, it can be proved that these limits exist and, correct to six decimal places, the values are

5 $$\left. \frac{d}{dx}(2^x) \right|_{x=0} \approx 0.693147 \qquad \left. \frac{d}{dx}(3^x) \right|_{x=0} \approx 1.098612$$

Thus, from Equation 4, we have

6 $$\frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Of all possible choices for the base a in Equation 4, the simplest differentiation formula occurs when $f'(0) = 1$. In view of the estimates of $f'(0)$ for $a = 2$ and $a = 3$, it seems reasonable that there is a number a between 2 and 3 for which $f'(0) = 1$. It is traditional to denote this value by the letter e. Thus we have the following definition.

7 DEFINITION OF THE NUMBER e

$$e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Geometrically this means that of all the possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at (0, 1) has a slope $f'(0)$ that is exactly 1.

(See Figures 10 and 11.) We call the function $f(x) = e^x$ the *natural exponential funct*

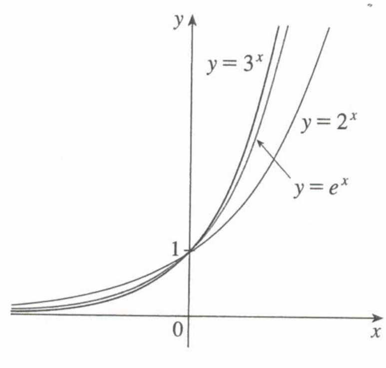


FIGURE 10

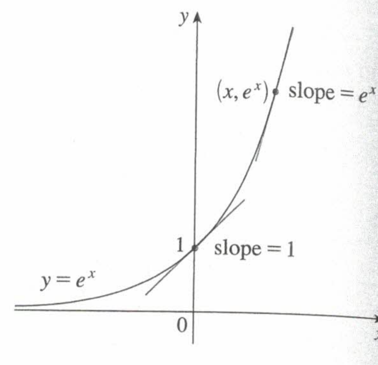


FIGURE 11

If we put $a = e$ and, therefore, $f'(0) = 1$ in Equation 4, it becomes the following im tant differentiation formula.

TEC Visual 7.2/7.3* uses the slope-a-scope to illustrate this formula.

8 DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

$$\frac{d}{dx}(e^x) = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivat The geometrical significance of this fact is that the slope of a tangent line to the cu $y = e^x$ at any point is equal to the y-coordinate of the point (see Figure 11).

V EXAMPLE 2 Differentiate the function $y = e^{\tan x}$.

SOLUTION To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \frac{du}{dx} = e^{\tan x} \sec^2 x$$

In general if we combine Formula 8 with the Chain Rule, as in Example 2, we get

9 $$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

EXAMPLE 3 Find y' if $y = e^{-4x} \sin 5x$.

SOLUTION Using Formula 9 and the Product Rule, we have

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5 \cos 5x - 4 \sin 5x)$$

We have seen that e is a number that lies somewhere between 2 and 3, but we use Equation 4 to estimate the numerical value of e more accurately. Let $e = 2^c$. T $e^x = 2^{cx}$. If $f(x) = 2^x$, then from Equation 4 we have $f'(x) = k2^x$, where the value of

$f'(0) \approx 0.693147$. Thus, by the Chain Rule,

$$e^x = \frac{d}{dx}(e^x) = \frac{d}{dx}(2^{cx}) = k2^{cx}\frac{d}{dx}(cx) = ck2^{cx}$$

Putting $x = 0$, we have $1 = ck$, so $c = 1/k$ and

$$e = 2^{1/k} \approx 2^{1/0.693147} \approx 2.71828$$

It can be shown that the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

EXAMPLE 4 In Example 6 in Section 3.7 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after t hours is

$$n = n_0 2^t$$

where n_0 is the initial population. Now we can use (4) and (5) to compute the growth rate:

■ The rate of growth is proportional to the size of the population.

$$\frac{dn}{dt} \approx n_0(0.693147)2^t$$

For instance, if the initial population is $n_0 = 1000$ cells, then the growth rate after two hours is

$$\left.\frac{dn}{dt}\right|_{t=2} \approx (1000)(0.693147)2^t\big|_{t=2}$$
$$= (4000)(0.693147) \approx 2773 \text{ cells/h}$$

□

V EXAMPLE 5 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

SOLUTION We differentiate to find any critical numbers:

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, when $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37$$

□

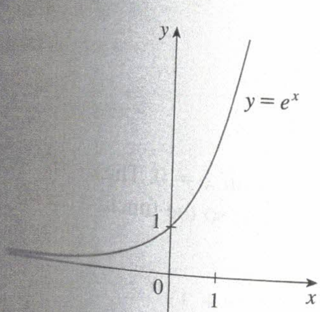


FIGURE 12
The natural exponential function

EXPONENTIAL GRAPHS

The exponential function $f(x) = e^x$ is one of the most frequently occurring functions in calculus and its applications, so it is important to be familiar with its graph (Figure 12) and properties. We summarize these properties as follows, using the fact that this function is just a special case of the exponential functions considered in Theorem 2 but with base $a = e > 1$.

10 **PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION** The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

EXAMPLE 6 Find $\displaystyle\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1}$.

SOLUTION We divide numerator and denominator by e^{2x}:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \lim_{x \to \infty} e^{-2x}}$$

$$= \frac{1}{1 + 0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0$$

EXAMPLE 7 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

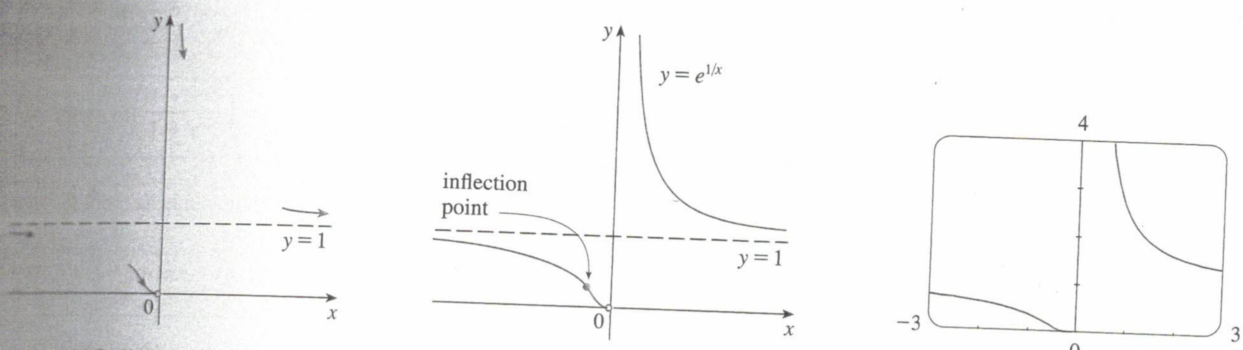
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 13(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 13(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 13(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

FIGURE 13

□

INTEGRATION

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

11

$$\int e^x \, dx = e^x + C$$

EXAMPLE 8 Evaluate $\int x^2 e^{x^3} \, dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\, dx$, so $x^2\, dx = \frac{1}{3}\, du$ and

$$\int x^2 e^{x^3}\, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C$$ □

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\, dx = -\frac{1}{3} e^{-3x} \Big]_0^1 = \frac{1}{3}(1 - e^{-3})$$ □

7.2 EXERCISES

1. (a) Write an equation that defines the exponential function with base $a > 0$.
(b) What is the domain of this function?
(c) If $a \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

4. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

5. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.3.

7. $y = 4^x - 3$

8. $y = 4^{x-3}$

9. $y = -2^{-x}$

10. $y = 1 + 2e^x$

11. $y = 1 - \frac{1}{2}e^{-x}$

12. $y = 2(1 - e^x)$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward
(b) shifting 2 units to the right
(c) reflecting about the x-axis
(d) reflecting about the y-axis
(e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$
(b) reflecting about the line $x = 2$

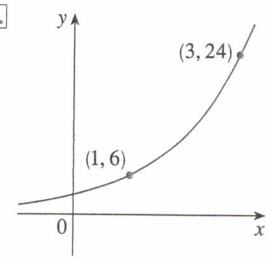
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1}{1 + e^x}$ (b) $f(x) = \dfrac{1}{1 - e^x}$

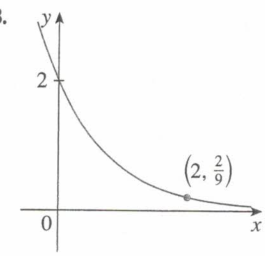
16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

17–18 Find the exponential function $f(x) = Ca^x$ whose ... given.

17.



18.



19. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are ... a coordinate grid where the unit of measurement is ... Show that, at a distance 2 ft to the right of the origi... height of the graph of f is 48 ft but the height of th... of g is about 265 mi.

20. Compare the rates of growth of the functions $f(x)$ ... $g(x) = 5^x$ by graphing both functions in several vie... angles. Find all points of intersection of the graphs ... one decimal place.

21. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ b... both f and g in several viewing rectangles. When d... graph of g finally surpass the graph of f?

22. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

23–30 Find the limit.

23. $\lim_{x \to \infty} (1.001)^x$

24. $\lim_{x \to -\infty} (1.001)^x$

25. $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

26. $\lim_{x \to \infty} e^{-x^2}$

27. $\lim_{x \to 2^+} e^{3/(2-x)}$

28. $\lim_{x \to 2^-} e^{3/(2-x)}$

29. $\lim_{x \to \infty} (e^{-2x} \cos x)$

30. $\lim_{x \to (\pi/2)^+} e^{\tan x}$

31–46 Differentiate the function.

31. $f(x) = (x^3 + 2x)e^x$

32. $y = \dfrac{e^x}{1 + x}$

33. $y = e^{ax^3}$

34. $y = e^u(\cos u + cu)$

35. $f(u) = e^{1/u}$

36. $g(x) = \sqrt{x}\, e^x$

37. $F(t) = e^{t \sin 2t}$

38. $f(t) = \sin(e^t) + e^{\sin t}$

39. $y = \sqrt{1 + 2e^{3x}}$

40. $y = e^{k \tan \sqrt{x}}$

41. $y = e^{e^x}$

42. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$

43. $y = \dfrac{ae^x + b}{ce^x + d}$

44. $y = \sqrt{1 + xe^{-2x}}$

45. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

46. $f(t) = \sin^2(e^{\sin^2 t})$

47–48 Find an equation of the tangent line to the curve at the given point.

47. $y = e^{2x} \cos \pi x, \quad (0, 1)$

48. $y = \dfrac{e^x}{x}, \quad (1, e)$

49. Find y' if $e^{x^2 y} = x + y$.

50. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

51. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

52. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

53. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

54. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

55. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

56. Find the thousandth derivative of $f(x) = xe^{-x}$.

57. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

58. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

59. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 10.4 we will see that this is a reasonable model for $p(t)$.]
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

60. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

61. Find the absolute maximum value of the function $f(x) = x - e^x$.

62. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

63–64 Find the absolute maximum and absolute minimum values of f on the given interval.

63. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

64. $f(x) = x^2 e^{-x/2}, \quad [-1, 6]$

65–66 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

65. $f(x) = (1 - x)e^{-x}$

66. $f(x) = \dfrac{e^x}{x^2}$

67–68 Discuss the curve using the guidelines of Section 4.5.

67. $y = e^{-1/(x+1)}$

68. $y = e^{-x} \sin x$

69. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

70–71 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

70. $f(x) = e^{\cos x}$ **71.** $f(x) = e^{x^3 - x}$

---

72. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

73–82 Evaluate the integral.

73. $\int_0^5 e^{-3x}\,dx$ **74.** $\int_0^1 xe^{-x^2}\,dx$

75. $\int e^x\sqrt{1 + e^x}\,dx$ **76.** $\int \frac{(1 + e^x)^2}{e^x}\,dx$

77. $\int (e^x + e^{-x})^2\,dx$ **78.** $\int e^x(4 + e^x)^5\,dx$

79. $\int \sin x\, e^{\cos x}\,dx$ **80.** $\int \frac{e^{1/x}}{x^2}\,dx$

81. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **82.** $\int e^x \sin(e^x)\,dx$

---

83. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

84. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

85. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

86. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

87. The **error function**

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.
(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\mathrm{erf}(b) - \mathrm{erf}(a)]$.
(b) Show that the function $y = e^{x^2}\mathrm{erf}(x)$ satisfies the ential equation $y' = 2xy + 2/\sqrt{\pi}$.

88. A bacteria population starts with 400 bacteria and gr rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How bacteria will there be after three hours?

89. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

90. Evaluate $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

91. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove

92. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b cha How does it change when a changes?

93. (a) Show that $e^x \geq 1 + x$ if $x \geq 0$. [*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasin for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leq \int_0^1 e^{x^2}\,dx \leq e$.

94. (a) Use the inequality of Exercise 93(a) to show that, f $x \geq 0$,

$$e^x \geq 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ giv Exercise 93(b).

95. (a) Use mathematical induction to prove that for $x \geq 0$ any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

7.3 LOGARITHMIC FUNCTIONS

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one. It therefore has an inverse function f^{-1}, which is called the **logarithmic function with base *a*** and is denoted by $\log_a$. If we use the formulation of an inverse function given by (7.1.3),

$$f^{-1}(x) = y \iff f(y) = x$$

then we have

1
$$\log_a x = y \iff a^y = x$$

Thus, if $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

EXAMPLE 1 Evaluate (a) $\log_3 81$, (b) $\log_{25} 5$, and (c) $\log_{10} 0.001$.

SOLUTION

(a) $\log_3 81 = 4$ because $3^4 = 81$

(b) $\log_{25} 5 = \frac{1}{2}$ because $25^{1/2} = 5$

(c) $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$ □

The cancellation equations (7.1.4), when applied to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, become

2
$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$
$$a^{\log_a x} = x \quad \text{for every } x > 0$$

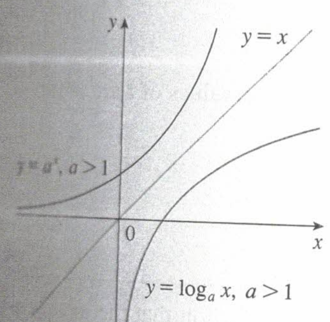


FIGURE 1

The logarithmic function $\log_a$ has domain $(0, \infty)$ and range $\mathbb{R}$ and is continuous since it is the inverse of a continuous function, namely, the exponential function. Its graph is the reflection of the graph of $y = a^x$ about the line $y = x$.

Figure 1 shows the case where $a > 1$. (The most important logarithmic functions have base $a > 1$.) The fact that $y = a^x$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

Figure 2 shows the graphs of $y = \log_a x$ with various values of the base a. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

The following theorem summarizes the properties of logarithmic functions.

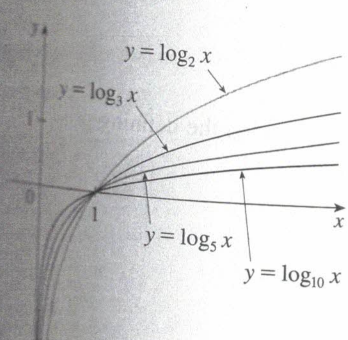


FIGURE 2

3 THEOREM If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r\log_a x$

Properties 1, 2, and 3 follow from the corresponding properties of exponential functions given in Section 7.2.

EXAMPLE 2 Use the properties of logarithms in Theorem 3 to evaluate the following.
(a) $\log_4 2 + \log_4 32$ (b) $\log_2 80 - \log_2 5$

SOLUTION
(a) Using Property 1 in Theorem 3, we have

$$\log_4 2 + \log_4 32 = \log_4(2 \cdot 32) = \log_4 64 = 3$$

since $4^3 = 64$.

(b) Using Property 2 we have

$$\log_2 80 - \log_2 5 = \log_2\left(\tfrac{80}{5}\right) = \log_2 16 = 4$$

since $2^4 = 16$.

The limits of exponential functions given in Section 7.2 are reflected in the following limits of logarithmic functions. (Compare with Figure 1.)

> **4** If $a > 1$, then
>
> $$\lim_{x \to \infty} \log_a x = \infty \qquad \text{and} \qquad \lim_{x \to 0^+} \log_a x = -\infty$$

In particular, the y-axis is a vertical asymptote of the curve $y = \log_a x$.

EXAMPLE 3 Find $\lim_{x \to 0} \log_{10}(\tan^2 x)$.

SOLUTION As $x \to 0$, we know that $t = \tan^2 x \to \tan^2 0 = 0$ and the values of t are positiv[e]. So by (4) with $a = 10 > 1$, we have

$$\lim_{x \to 0} \log_{10}(\tan^2 x) = \lim_{t \to 0^+} \log_{10} t = -\infty$$

NATURAL LOGARITHMS

NOTATION FOR LOGARITHMS

Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

Of all possible bases a for logarithms, we will see in the next section that the most co[nvenient] venient choice of a base is the number e, which was defined in Section 7.2. The logarit[hm] with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

If we put $a = e$ and replace $\log_e$ with "ln" in (1) and (2), then the defining proper[ties] of the natural logarithm function become

5
$$\ln x = y \iff e^y = x$$

6
$$\ln(e^x) = x \qquad x \in \mathbb{R}$$
$$e^{\ln x} = x \qquad x > 0$$

In particular, if we set $x = 1$, we get

$$\ln e = 1$$

EXAMPLE 4 Find x if $\ln x = 5$.

SOLUTION 1 From (5) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

(If you have trouble working with the "ln" notation, just replace it by $\log_e$. Then the equation becomes $\log_e x = 5$; so, by the definition of logarithm, $e^5 = x$.)

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But the second cancellation equation in (6) says that $e^{\ln x} = x$. Therefore $x = e^5$. □

V EXAMPLE 5 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (6):

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approximate the solution: to four decimal places, $x \approx 0.8991$. □

V EXAMPLE 6 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Properties 3 and 1 of logarithms, we have

$$\ln a + \tfrac{1}{2}\ln b = \ln a + \ln b^{1/2}$$

$$= \ln a + \ln\sqrt{b}$$

$$= \ln\left(a\sqrt{b}\right)$$

□

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

7 CHANGE OF BASE FORMULA For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}$$

PROOF Let $y = \log_a x$. Then, from (1), we have $a^y = x$. Taking natural logarithms sides of this equation, we get $y \ln a = \ln x$. Therefore

$$y = \frac{\ln x}{\ln a}$$

Scientific calculators have a key for natural logarithms, so Formula 7 enables us a calculator to compute a logarithm with any base (as shown in the following exa Similarly, Formula 7 allows us to graph any logarithmic function on a graphing cal or computer (see Exercises 20–22).

EXAMPLE 7 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 7 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$

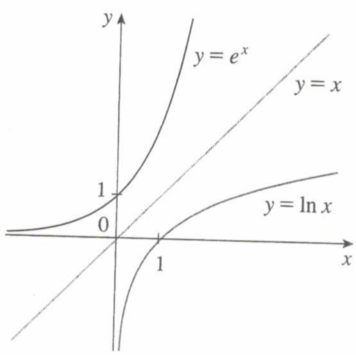


FIGURE 3

The graphs of the exponential function $y = e^x$ and its inverse function, the natur arithm function, are shown in Figure 3. Because the curve $y = e^x$ crosses the y-axi a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the x-axis with a slop

In common with all other logarithmic functions with base greater than 1, the n logarithm is a continuous, increasing function defined on $(0, \infty)$ and the y-axis is a v asymptote.

If we put $a = e$ in (4), then we have the following limits:

8 $$\lim_{x \to \infty} \ln x = \infty \qquad \lim_{x \to 0^+} \ln x = -\infty$$

V EXAMPLE 8 Sketch the graph of the function $y = \ln(x - 2) - 1$.

SOLUTION We start with the graph of $y = \ln x$ as given in Figure 3. Using the transfor tions of Section 1.3, we shift it 2 units to the right to get the graph of $y = \ln(x - 2)$ then we shift it 1 unit downward to get the graph of $y = \ln(x - 2) - 1$. (See Figure Notice that the line $x = 2$ is a vertical asymptote since

$$\lim_{x \to 2^+} [\ln(x - 2) - 1] = -\infty$$

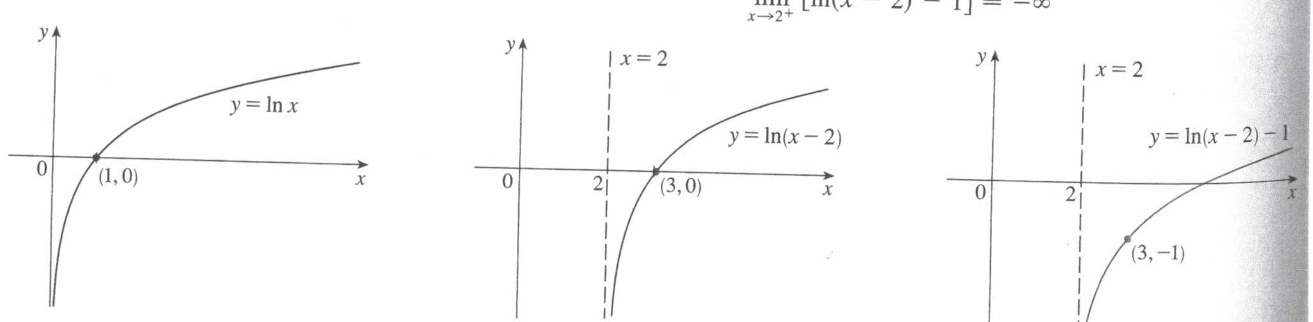


FIGURE 4

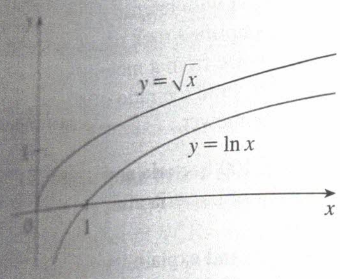


FIGURE 5

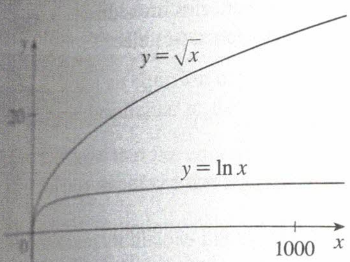


FIGURE 6

We have seen that $\ln x \to \infty$ as $x \to \infty$. But this happens *very* slowly. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 5 and 6.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\dfrac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm. In fact, we will be able to show in Section 7.8 that

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$$

for any positive power p. So for large x, the values of $\ln x$ are very small compared with x^p. (See Exercise 70.)

7.3 EXERCISES

1. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

2. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_5 125$ (b) $\log_3 \frac{1}{27}$

4. (a) $\ln(1/e)$ (b) $\log_{10} \sqrt{10}$

5. (a) $\log_5 \frac{1}{25}$ (b) $e^{\ln 15}$

6. (a) $\log_{10} 0.1$ (b) $\log_8 320 - \log_8 5$

7. (a) $\log_2 6 - \log_2 15 + \log_2 20$
(b) $\log_3 100 - \log_3 18 - \log_3 50$

8. (a) $e^{-2\ln 5}$ (b) $\ln(\ln e^{e^{10}})$

9–12 Use the properties of logarithms to expand the quantity.

9. $\log_2\left(\dfrac{x^3 y}{z^2}\right)$

10. $\ln \sqrt{a(b^2 + c^2)}$

11. $\ln (uv)^{10}$

12. $\ln \dfrac{3x^2}{(x+1)^5}$

13–18 Express the quantity as a single logarithm.

13. $\log_{10} a - \log_{10} b + \log_{10} c$

14. $\ln(x + y) + \ln(x - y) - 2 \ln z$

15. $\ln 5 + 5 \ln 3$

16. $\ln 3 + \frac{1}{3} \ln 8$

17. $\ln(1 + x^2) + \frac{1}{2} \ln x - \ln \sin x$

18. $\ln(a + b) + \ln(a - b) - 2 \ln c$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x, \quad y = \log_4 x, \quad y = \log_6 x, \quad y = \log_8 x$

21. $y = \log_{1.5} x, \quad y = \ln x, \quad y = \log_{10} x, \quad y = \log_{50} x$

22. $y = \ln x, \quad y = \log_{10} x, \quad y = e^x, \quad y = 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

25–34 Solve each equation for x.

25. (a) $2 \ln x = 1$ (b) $e^{-x} = 5$

26. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

27. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

28. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

29. $3xe^x + x^2e^x = 0$

30. $10(1 + e^{-x})^{-1} = 3$

31. $\ln(\ln x) = 1$

32. $e^{e^x} = 10$

33. $e^{2x} - e^x - 6 = 0$

34. $\ln(2x + 1) = 2 - \ln x$

35–36 Find the solution of the equation correct to four decimal places.

35. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

36. (a) $\ln(1 + \sqrt{x}) = 2$ (b) $3^{1/(x-4)} = 7$

37–38 Solve each inequality for x.

37. (a) $e^x < 10$ (b) $\ln x > -1$

38. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

39. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

40. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.
(c) At what time is the velocity equal to half the initial velocity?

41. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

42. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is th defined to be $L = 10 \log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driv lawn mower is measured at 106 dB. Find the ratio of the i sity of the rock music to that of the mower.

43. If a bacteria population starts with 100 bacteria and doubl every three hours, then the number of bacteria after t hour is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meani
(b) When will the population reach 50,000?

44. When a camera flash goes off, the batteries immediately b to recharge the flash's capacitor, which stores electric cha given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured i seconds.)
(a) Find the inverse of this function and explain its mean
(b) How long does it take to recharge the capacitor to 90 capacity if $a = 2$?

45–50 Find the limit.

45. $\lim_{x\to3^+} \ln(x^2 - 9)$

46. $\lim_{x\to2^-} \log_5(8x - x^4)$

47. $\lim_{x\to0} \ln(\cos x)$

48. $\lim_{x\to0^+} \ln(\sin x)$

49. $\lim_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)]$

50. $\lim_{x\to\infty} [\ln(2 + x) - \ln(1 + x)]$

51–52 Find the domain of the function.

51. $f(x) = \log_{10}(x^2 - 9)$

52. $f(x) = \ln x + \ln(2$

53–54 Find (a) the domain of f and (b) f^{-1} and its domair

53. $f(x) = \sqrt{3 - e^{2x}}$

54. $f(x) = \ln(2 + \ln x$

55–60 Find the inverse function.

55. $y = \ln(x + 3)$

56. $y = 2^{10^x}$

57. $f(x) = e^{x^3}$

58. $y = (\ln x)^2, \quad x \geqslant$

59. $y = \log_{10}\left(1 + \dfrac{1}{x}\right)$

60. $y = \dfrac{e^x}{1 + 2e^x}$

61. On what interval is the function $f(x) = e^{3x} - e^x$ increa

62. On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

63. On what intervals is the curve $y = (x^2 - 2)e^{-x}$ concave upward?

64. For the period from 1980 to 2000, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{85}{1 + 53e^{-0.5t}}$$

where the time t is measured in years since midyear 1980, so $0 \leq t \leq 20$. Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

65. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is an odd function.
(b) Find the inverse function of f.

66. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

67. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?

68. Any function of the form $f(x) = [g(x)]^{h(x)}$, where $g(x) > 0$, can be analyzed as a power of e by writing $g(x) = e^{\ln g(x)}$ so that $f(x) = e^{h(x) \ln g(x)}$. Using this device, calculate each limit.
(a) $\lim_{x \to \infty} x^{\ln x}$
(b) $\lim_{x \to 0^+} x^{-\ln x}$
(c) $\lim_{x \to 0^+} x^{1/x}$
(d) $\lim_{x \to \infty} (\ln 2x)^{-\ln x}$

69. Let $a > 1$. Prove, using Definitions 4.4.6 and 4.4.7, that
(a) $\lim_{x \to -\infty} a^x = 0$
(b) $\lim_{x \to \infty} a^x = \infty$

70. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if } x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

71. Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

72. A **prime number** is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... We denote by $\pi(n)$ the number of primes that are less than or equal to n. For instance, $\pi(15) = 6$ because there are six primes smaller than 15.
(a) Calculate the numbers $\pi(25)$ and $\pi(100)$.
[*Hint:* To find $\pi(100)$, first compile a list of the primes up to 100 using the *sieve of Eratosthenes:* Write the numbers from 2 to 100 and cross out all multiples of 2. Then cross out all multiples of 3. The next remaining number is 5, so cross out all remaining multiples of it, and so on.]
(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to n is approximately $n/\ln n$ when n is large. More precisely, he conjectured that

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1$$

This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the **Prime Number Theorem**. Provide evidence for the truth of this theorem by computing the ratio of $\pi(n)$ to $n/\ln n$ for $n = 100, 1000, 10^4, 10^5, 10^6$, and 10^7. Use the following data: $\pi(1000) = 168$, $\pi(10^4) = 1229$, $\pi(10^5) = 9592$, $\pi(10^6) = 78{,}498$, $\pi(10^7) = 664{,}579$.
(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.

7.4 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In this section we find the derivatives of the logarithmic functions $y = \log_a x$ and the exponential functions $y = a^x$. We start with the natural logarithmic function $y = \ln x$. We know that it is differentiable because it is the inverse of the differentiable function $y = e^x$.

1

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

PROOF Let $y = \ln x$. Then

$$e^y = x$$

Differentiating this equation implicitly with respect to x, we get

$$e^y \frac{dy}{dx} = 1$$

and so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3+1}(3x^2) = \frac{3x^2}{x^3+1}$$

In general, if we combine Formula 1 with the Chain Rule as in Example 1, we get

2 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 2 Find $\frac{d}{dx}\ln(\sin x)$.

SOLUTION Using (2), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 3 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 4 Find $\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)(\frac{1}{2})(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

■ Figure 1 shows the graph of the function f of Example 4 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

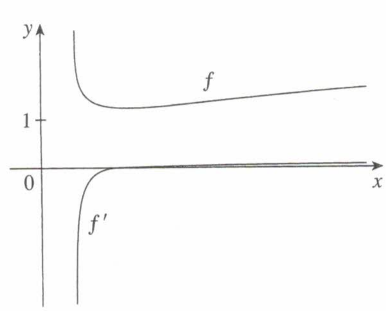


FIGURE 1

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}[\ln(x+1) - \tfrac{1}{2}\ln(x-2)] = \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) □

EXAMPLE 5 Find the absolute minimum value of $f(x) = x^2 \ln x$.

SOLUTION The domain is $(0, \infty)$ and the Product Rule gives

$$f'(x) = x^2 \cdot \frac{1}{x} + 2x \ln x = x(1 + 2 \ln x)$$

Therefore $f'(x) = 0$ when $2 \ln x = -1$, that is, $\ln x = -\frac{1}{2}$, or $x = e^{-1/2}$. Also, $f'(x) > 0$ when $x > e^{-1/2}$ and $f'(x) < 0$ for $0 < x < e^{-1/2}$. So, by the First Derivative Test for Absolute Extreme Values, $f(1/\sqrt{e}) = -1/(2e)$ is the absolute minimum. □

EXAMPLE 6 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 4.5.

SOLUTION

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = \log_e 1 = 0$ (since $e^0 = 1$), so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \text{and} \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

by (7.3.8). Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.

$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

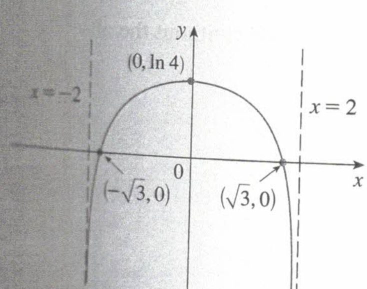


FIGURE 2
$y = \ln(4 - x^2)$

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 2. □

V EXAMPLE 7 Find $f'(x)$ if $f(x) = \ln |x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 7 is worth remembering:

3
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

The corresponding integration formula is

4
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case ($n = -1$) is supplied by Formula 4.

EXAMPLE 8 Find, correct to three decimal places, the area of the region under the hyperbola $xy = 1$ from $x = 1$ to $x = 2$.

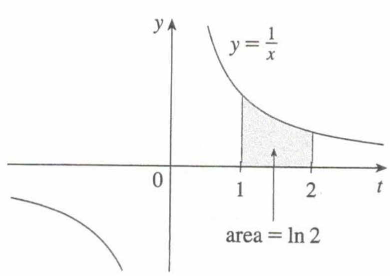


FIGURE 3

SOLUTION The given region is shown in Figure 3. Using Formula 4 (without the absolu value sign, since $x > 0$), we see that the area is

$$A = \int_1^2 \frac{1}{x}\,dx = \ln x\Big]_1^2$$
$$= \ln 2 - \ln 1 = \ln 2 \approx 0.693$$

EXAMPLE 9 Evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\,dx$ o (except for the constant factor 2). Thus $x\,dx = \frac{1}{2}\,du$ and

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$
$$= \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the properties of logarithms to write the answer as

$$\ln \sqrt{x^2 + 1} + C$$

but this isn't necessary. □

Since the function $f(x) = (\ln x)/x$ in Example 10 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 4.

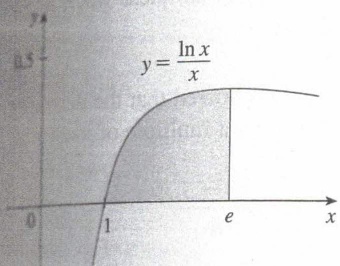


FIGURE 4

V EXAMPLE 10 Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\bigg]_0^1 = \frac{1}{2}$$ □

V EXAMPLE 11 Calculate $\int \tan x\,dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

This suggests that we should substitute $u = \cos x$ since then $du = -\sin x\,dx$ and so $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$ □

Since $-\ln|\cos x| = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 11 can also be written as

5 $$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$

GENERAL LOGARITHMIC AND EXPONENTIAL FUNCTIONS

Formula 7 in Section 7.3 expresses a logarithmic function with base a in terms of the natural logarithmic function:

$$\log_a x = \frac{\ln x}{\ln a}$$

Since $\ln a$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

6
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

EXAMPLE 12 Using Formula 6 and the Chain Rule, we get

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x) = \frac{\cos x}{(2 + \sin x)\ln 10}$$

From Formula 6 we see one of the main reasons that natural logarithms (lo with base e) are used in calculus: The differentiation formula is simplest whe because $\ln e = 1$.

EXPONENTIAL FUNCTIONS WITH BASE a In Section 7.2 we showed that the de of the general exponential function $f(x) = a^x$, $a > 0$, is a constant multiple of its

$$f'(x) = f'(0)a^x \qquad \text{where} \qquad f'(0) = \lim_{h \to 0}\frac{a^h - 1}{h}$$

We are now in a position to show that the value of the constant is $f'(0) = \ln a$.

7
$$\frac{d}{dx}(a^x) = a^x \ln a$$

PROOF We use the fact that $e^{\ln a} = a$:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$
$$= (e^{\ln a})^x(\ln a) = a^x \ln a$$

In Example 6 in Section 3.7 we considered a population of bacteria cells that d every hour and we saw that the population after t hours is $n = n_0 2^t$, where n_0 is the population. Formula 7 enables us to find the growth rate:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

EXAMPLE 13 Combining Formula 7 with the Chain Rule, we have

$$\frac{d}{dx}\left(10^{x^2}\right) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2 \ln 10)x10^{x^2}$$

The integration formula that follows from Formula 7 is

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad a \neq 1$$

EXAMPLE 14 $$\int_0^5 2^x\,dx = \frac{2^x}{\ln 2}\bigg]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$$ □

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 15 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

If we hadn't used logarithmic differentiation in Example 15, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ □

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 3. We illustrate this procedure by proving the general version of the Power Rule, as promised in Section 3.3.

THE POWER RULE If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

⊘ You should distinguish carefully between the Power Rule $[(d/dx)\, x^n = nx^{n-1}$ the base is variable and the exponent is constant, and the rule for differentiating e tial functions $[(d/dx)\, a^x = a^x \ln a]$, where the base is constant and the exponent is

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0$ (a and b are constants)
2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$
3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$
4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the n example.

V EXAMPLE 16 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

■ Figure 5 illustrates Example 16 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

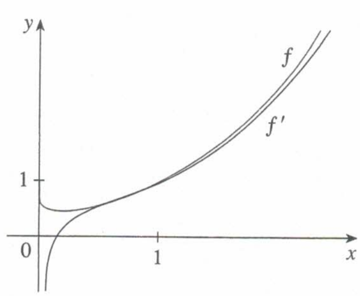


FIGURE 5

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\ln x}\right) = e^{\sqrt{x}\ln x}\frac{d}{dx}\left(\sqrt{x}\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

THE NUMBER e AS A LIMIT

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$

8

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

Formula 8 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 6 and a table of values for small values of x.

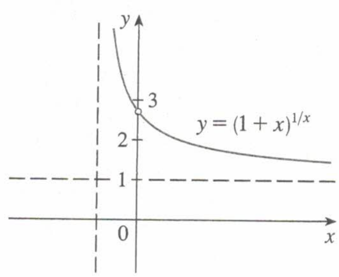


FIGURE 6

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

7.4 EXERCISES

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–26 Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \log_2(1 - 3x)$

6. $f(x) = \log_5(xe^x)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln \sqrt[5]{x}$

9. $f(x) = \sin x \ln(5x)$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

14. $F(y) = y \ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln(x^4 \sin^2 x)$

17. $h(t) = t^3 - 3^t$

18. $y = 10^{\tan \theta}$

19. $y = \ln |2 - x - 5x^2|$

20. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $y = [\ln(1 + e^x)]^2$

23. $y = 2x \log_{10} \sqrt{x}$

24. $y = \log_2(e^{-x} \cos \pi x)$

25. $y = 5^{-1/x}$

26. $y = 2^{3^{x^2}}$

27–30 Find y' and y''.

27. $y = x^2 \ln(2x)$

28. $y = (\ln x)/x^2$

29. $y = \ln(x + \sqrt{1 + x^2})$

30. $y = \ln(\sec x + \tan x)$

31–34 Differentiate f and find the domain of f.

31. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

32. $f(x) = \dfrac{1}{1 + \ln x}$

33. $f(x) = \ln(x^2 - 2x)$

34. $f(x) = \ln \ln \ln x$

35. If $f(x) = \dfrac{\ln x}{1 + x^2}$, find $f'(1)$.

36. If $f(x) = \ln(1 + e^{2x})$, find $f'(0)$.

37–38 Find an equation of the tangent line to the curve at the given point.

37. $y = \ln(xe^{x^2})$, $(1, 1)$

38. $y = \ln(x^3 - 7)$, $(2, 0)$

39. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

40. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

41–52 Use logarithmic differentiation to find the derivative of the function.

41. $y = (2x + 1)^5(x^4 - 3)^6$

42. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

43. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

44. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

45. $y = x^x$

46. $y = x^{\cos x}$

47. $y = x^{\sin x}$

48. $y = \sqrt{x}^{\,x}$

49. $y = (\cos x)^x$

50. $y = (\sin x)^{\ln x}$

51. $y = (\tan x)^{1/x}$

52. $y = (\ln x)^{\cos x}$

53. Find y' if $y = \ln(x^2 + y^2)$.

54. Find y' if $x^y = y^x$.

55. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

56. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

57–58 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

57. $(x - 4)^2 = \ln x$

58. $\ln(4 - x^2) = x$

59. Find the intervals of concavity and the inflection points of the function $f(x) = (\ln x)/\sqrt{x}$.

60. Find the absolute minimum value of the function $f(x) = x \ln x$.

61–64 Discuss the curve under the guidelines of Section 4.5.

61. $y = \ln(\sin x)$

62. $y = \ln(\tan^2 x)$

63. $y = \ln(1 + x^2)$

64. $y = \ln(x^2 - 3x + 2)$

CAS **65.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

66. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

67. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.

(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

68. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.

(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).

(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

69–80 Evaluate the integral.

69. $\int_2^4 \frac{3}{x}\,dx$

70. $\int_1^2 \frac{4+u^2}{u^3}\,du$

71. $\int_1^2 \frac{dt}{8-3t}$

72. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

73. $\int_1^e \frac{x^2+x+1}{x}\,dx$

74. $\int \frac{\sin(\ln x)}{x}\,dx$

75. $\int \frac{(\ln x)^2}{x}\,dx$

76. $\int \frac{\cos x}{2+\sin x}\,dx$

77. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

78. $\int \frac{e^x}{e^x+1}\,dx$

79. $\int_1^2 10^t\,dt$

80. $\int x2^{x^2}\,dx$

---

81. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

82. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x-2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

83. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

84. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

85. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

86. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

87. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

88. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

89. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

90. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

91. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

92. Show that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

7.2* THE NATURAL LOGARITHMIC FUNCTION

If your instructor has assigned Sections 7.2–7.4, you need not read Sections 7.2*, 7.3*, and 7.4* (pp. 421–446).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

1 DEFINITION The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under

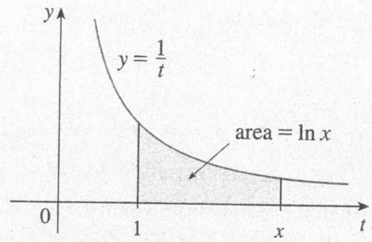


FIGURE 1

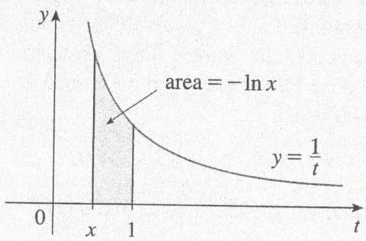


FIGURE 2

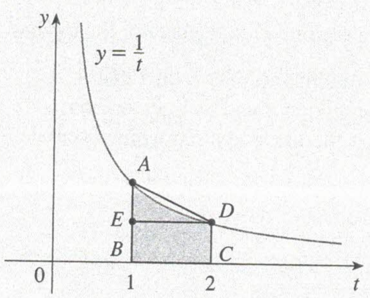


FIGURE 3

the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

For $0 < x < 1$,

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shown in Figure 2.

EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.

(b) Use the Midpoint Rule with $n = 10$ to estimate the value of ln 2.

SOLUTION

(a) We can interpret ln 2 as the area under the curve $y = 1/t$ from 1 to 2. From Fig we see that this area is larger than the area of rectangle *BCDE* and smaller than the of trapezoid *ABCD*. Thus we have

$$\frac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$\frac{1}{2} < \ln 2 < \frac{3}{4}$$

(b) If we use the Midpoint Rule with $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.1$, we get

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx (0.1)[f(1.05) + f(1.15) + \cdots + f(1.95)]$$

$$= (0.1)\left(\frac{1}{1.05} + \frac{1}{1.15} + \cdots + \frac{1}{1.95}\right) \approx 0.693$$

Notice that the integral that defines $\ln x$ is exactly the type of integral discussed in of the Fundamental Theorem of Calculus (see Section 5.3). In fact, using that the we have

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

and so

2
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

We now use this differentiation rule to prove the following properties of the loga function.

3 LAWS OF LOGARITHMS If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r \ln x$

PROOF

1. Let $f(x) = \ln(ax)$, where a is a positive constant. Then, using Equation 2 and the Chain Rule, we have

$$f'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Therefore $f(x)$ and $\ln x$ have the same derivative and so they must differ by a constant:

$$\ln(ax) = \ln x + C$$

Putting $x = 1$ in this equation, we get $\ln a = \ln 1 + C = 0 + C = C$. Thus

$$\ln(ax) = \ln x + \ln a$$

If we now replace the constant a by any number y, we have

$$\ln(xy) = \ln x + \ln y$$

2. Using Law 1 with $x = 1/y$, we have

$$\ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y} \cdot y\right) = \ln 1 = 0$$

and so $$\ln\frac{1}{y} = -\ln y$$

Using Law 1 again, we have

$$\ln\left(\frac{x}{y}\right) = \ln\left(x \cdot \frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y$$

The proof of Law 3 is left as an exercise. □

EXAMPLE 2 Expand the expression $\ln\dfrac{(x^2+5)^4 \sin x}{x^3+1}$.

SOLUTION Using Laws 1, 2, and 3, we get

$$\begin{aligned}\ln\frac{(x^2+5)^4 \sin x}{x^3+1} &= \ln(x^2+5)^4 + \ln\sin x - \ln(x^3+1)\\ &= 4\ln(x^2+5) + \ln\sin x - \ln(x^3+1)\end{aligned}$$ □

V EXAMPLE 3 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\ln a + \tfrac{1}{2}\ln b = \ln a + \ln b^{1/2} = \ln a + \ln\sqrt{b} = \ln\left(a\sqrt{b}\right)$$ □

In order to graph $y = \ln x$, we first determine its limits:

4 (a) $\displaystyle\lim_{x\to\infty} \ln x = \infty$ (b) $\displaystyle\lim_{x\to 0^+} \ln x = -\infty$

PROOF

(a) Using Law 3 with $x = 2$ and $r = n$ (where n is any positive integer), we have $\ln(2^n) = n \ln 2$. Now $\ln 2 > 0$, so this shows that $\ln(2^n) \to \infty$ as $n \to \infty$. But $\ln x$ is an increasing function since its derivative $1/x$ is positive. Therefore $\ln x \to \infty$ as $x \to \infty$.

(b) If we let $t = 1/x$, then $t \to \infty$ as $x \to 0^+$. Thus, using (a), we have

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln\left(\frac{1}{t}\right) = \lim_{t \to \infty} (-\ln t) = -\infty$$

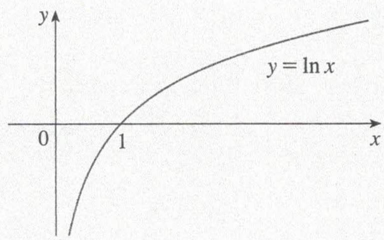


FIGURE 4

If $y = \ln x$, $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} > 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$$

which shows that $\ln x$ is increasing and concave downward on $(0, \infty)$. Putting this information together with (4), we draw the graph of $y = \ln x$ in Figure 4.

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where $\ln x$ takes on the value 1. (See Figure 5.) This important number is denoted by e.

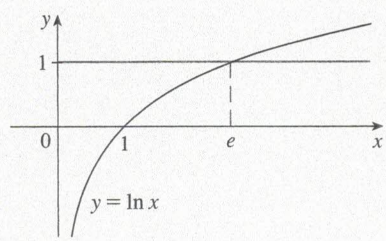


FIGURE 5

5 DEFINITION e is the number such that $\ln e = 1$.

EXAMPLE 4 Use a graphing calculator or computer to estimate the value of e.

SOLUTION According to Definition 5, we estimate the value of e by graphing the curves $y = \ln x$ and $y = 1$ and determining the x-coordinate of the point of intersection. By zooming in repeatedly, as in Figure 6, we find that

$$e \approx 2.718$$

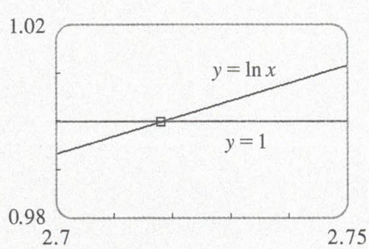


FIGURE 6

With more sophisticated methods, it can be shown that the approximate value of e, to 20 decimal places, is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

Now let's use Formula 2 to differentiate functions that involve the natural logarithm function.

V EXAMPLE 5 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$

In general, if we combine Formula 2 with the Chain Rule as in Example 5, we get

$$\boxed{6} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

V EXAMPLE 6 Find $\frac{d}{dx}\ln(\sin x)$.

SOLUTION Using (6), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$ □

EXAMPLE 7 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$ □

▲ Figure 7 shows the graph of the function f of Example 8 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

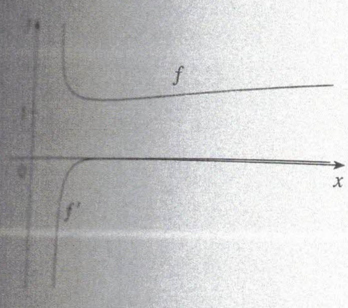


FIGURE 7

EXAMPLE 8 Find $\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\frac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)}$$

SOLUTION 2 If we first simplify the given function using the laws of logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right] = \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.) □

EXAMPLE 9 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 4.5.

SOLUTION

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x\to 2^-}\ln(4 - x^2) = -\infty \qquad \text{and} \qquad \lim_{x\to -2^+}\ln(4 - x^2) = -\infty$$

by (4). Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

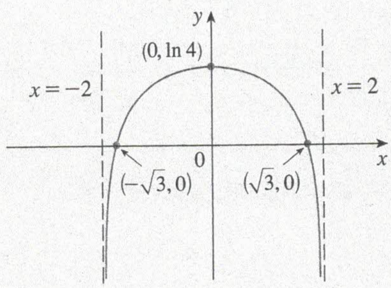


FIGURE 8
$y = \ln(4 - x^2)$

E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increas
on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at
$f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no
inflection point.

H. Using this information, we sketch the curve in Figure 8.

V EXAMPLE 10 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$.

The result of Example 10 is worth remembering:

7
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

The corresponding integration formula is

8
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case ($n = -1$) is supplied by Formula 8.

V EXAMPLE 11 Evaluate $\displaystyle\int \frac{x}{x^2 + 1}\, dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\, dx$ occurs (except for the constant factor 2). Thus $x\, dx = \frac{1}{2}\, du$ and

$$\int \frac{x}{x^2 + 1}\, dx = \tfrac{1}{2} \int \frac{du}{u} = \tfrac{1}{2} \ln|u| + C$$

$$= \tfrac{1}{2} \ln|x^2 + 1| + C = \tfrac{1}{2} \ln(x^2 + 1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the properties of logarithms to write the answer as

$$\ln \sqrt{x^2 + 1} + C$$

but this isn't necessary. □

■ Since the function $f(x) = (\ln x)/x$ in Example 12 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 9.

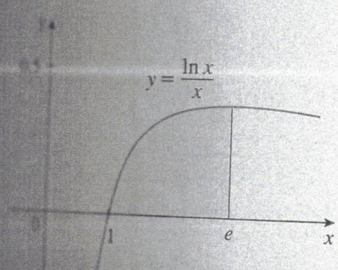


FIGURE 9

V EXAMPLE 12 Calculate $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\, dx = \int_0^1 u\, du = \frac{u^2}{2}\bigg]_0^1 = \frac{1}{2}$$ □

V EXAMPLE 13 Calculate $\displaystyle\int \tan x\, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$ since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u}$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$ □

Since $-\ln|\cos x| = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 13 can also be written as

9 $$\boxed{\int \tan x\, dx = \ln|\sec x| + C}$$

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotien powers can often be simplified by taking logarithms. The method used in the follo example is called **logarithmic differentiation**.

V EXAMPLE 14 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Loga-rithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

■ If we hadn't used logarithmic differentiation in Example 14, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Law of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can $|y| = |f(x)|$ and use Equation 7.

7.2* EXERCISES

1–4 Use the Laws of Logarithms to expand the quantity.

1. $\ln \dfrac{r^2}{3\sqrt{s}}$

2. $\ln \sqrt{a(b^2+c^2)}$

3. $\ln (uv)^{10}$

4. $\ln \dfrac{3x^2}{(x+1)^5}$

5–8 Express the quantity as a single logarithm.

5. $\ln 5 + 5\ln 3$

6. $\ln 3 + \frac{1}{3}\ln 8$

7. $\ln(1+x^2) + \frac{1}{2}\ln x - \ln\sin x$

8. $\ln(a+b) + \ln(a-b) - 2\ln c$

9–12 Make a rough sketch of the graph of each function. Do n use a calculator. Just use the graph given in Figure 4 and, if ne sary, the transformations of Section 1.3.

9. $y = -\ln x$

10. $y = \ln|x|$

11. $y = \ln(x+3)$

12. $y = 1 + \ln(x-2)$

13–14 Find the limit.

13. $\lim_{x \to 3^+} \ln(x^2 - 9)$

14. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

15–34 Differentiate the function.

15. $f(x) = \sqrt{x} \ln x$

16. $f(x) = \ln(x^2 + 10)$

17. $f(x) = \sin(\ln x)$

18. $f(x) = \ln(\sin^2 x)$

19. $f(x) = \sqrt[5]{\ln x}$

20. $f(x) = \ln \sqrt[5]{x}$

21. $f(x) = \sin x \ln(5x)$

22. $h(x) = \ln(x + \sqrt{x^2 - 1})$

23. $g(x) = \ln \dfrac{a - x}{a + x}$

24. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

25. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

26. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

27. $g(x) = \ln(x\sqrt{x^2 - 1})$

28. $y = \ln(x^4 \sin^2 x)$

29. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

30. $y = (\ln \tan x)^2$

31. $y = \ln|2 - x - 5x^2|$

32. $y = \ln \tan^2 x$

33. $y = \tan[\ln(ax + b)]$

34. $y = \ln|\cos(\ln x)|$

35–36 Find y' and y''.

35. $y = x^2 \ln(2x)$

36. $y = \ln(\sec x + \tan x)$

37–40 Differentiate f and find the domain of f.

37. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

38. $f(x) = \ln(x^2 - 2x)$

39. $f(x) = \sqrt{1 - \ln x}$

40. $f(x) = \ln \ln \ln x$

---

41. If $f(x) = \dfrac{\ln x}{1 + x^2}$, find $f'(1)$.

42. If $f(x) = \dfrac{\ln x}{x}$, find $f''(e)$.

43–44 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

43. $f(x) = \sin x + \ln x$

44. $f(x) = \ln(x^2 + x + 1)$

45–46 Find an equation of the tangent line to the curve at the given point.

45. $y = \sin(2 \ln x)$, $(1, 0)$

46. $y = \ln(x^3 - 7)$, $(2, 0)$

---

47. Find y' if $y = \ln(x^2 + y^2)$.

48. Find y' if $\ln xy = y \sin x$.

49. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

50. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

51–52 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

51. $(x - 4)^2 = \ln x$

52. $\ln(4 - x^2) = x$

53–56 Discuss the curve under the guidelines of Section 4.5.

53. $y = \ln(\sin x)$

54. $y = \ln(\tan^2 x)$

55. $y = \ln(1 + x^2)$

56. $y = \ln(x^2 - 3x + 2)$

CAS **57.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

58. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

59–62 Use logarithmic differentiation to find the derivative of the function.

59. $y = (2x + 1)^5(x^4 - 3)^6$

60. $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{\sqrt[3]{x}}$

61. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

62. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

63–72 Evaluate the integral.

63. $\displaystyle\int_2^4 \frac{3}{x}\,dx$

64. $\displaystyle\int_1^2 \frac{4 + u^2}{u^3}\,du$

65. $\displaystyle\int_1^2 \frac{dt}{8 - 3t}$

66. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

67. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\,dx$

68. $\displaystyle\int_e^6 \frac{dx}{x \ln x}$

69. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

70. $\displaystyle\int \frac{\cos x}{2 + \sin x}\,dx$

71. $\displaystyle\int \frac{\sin 2x}{1 + \cos^2 x}\,dx$

72. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

---

73. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 13.

74. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x - 2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

75. Find the volume of the solid obtained by rotating the region under the curve $y = 1/\sqrt{x+1}$ from 0 to 1 about the x-axis.

76. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2 + 1}$$

from 0 to 3 about the y-axis.

77. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

78. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

79. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

80. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

81. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate ln 1.5.

82. Refer to Example 1.
(a) Find an equation of the tangent line to the curve $y =$ that is parallel to the secant line AD.
(b) Use part (a) to show that $\ln 2 > 0.66$.

83. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{}{n}$$

84. Prove the third law of logarithms. [*Hint:* Start by show that both sides of the equation have the same derivative.

85. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the r

86. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several view rectangles. When does the graph of f finally surpass graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing r angle that displays the behavior of the function as x
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

87. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

7.3* THE NATURAL EXPONENTIAL FUNCTION

Since ln is an increasing function, it is one-to-one and therefore has an inverse f which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

1

$$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

2

$$\exp(\ln x) = x \quad \text{and} \quad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$

$$\exp(1) = e \quad \text{since} \quad \ln e = 1$$

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ abou

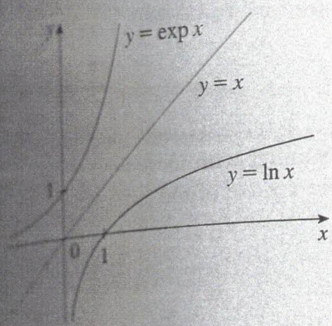


FIGURE 1

$y = x$. (See Figure 1.) The domain of exp is the range of ln, that is, $(-\infty, \infty)$; the range of exp is the domain of ln, that is, $(0, \infty)$.

If r is any rational number, then the third law of logarithms gives

$$\ln(e^r) = r \ln e = r$$

Therefore, by (1), $$\exp(r) = e^r$$

Thus $\exp(x) = e^x$ whenever x is a rational number. This leads us to define e^x, even for irrational values of x, by the equation

$$\boxed{e^x = \exp(x)}$$

In other words, for the reasons given, we define e^x to be the inverse of the function $\ln x$. In this notation (1) becomes

3 $$\boxed{e^x = y \iff \ln y = x}$$

and the cancellation equations (2) become

4 $$\boxed{e^{\ln x} = x \qquad x > 0}$$

5 $$\boxed{\ln(e^x) = x \qquad \text{for all } x}$$

EXAMPLE 1 Find x if $\ln x = 5$.

SOLUTION 1 From (3) we see that

$$\ln x = 5 \qquad \text{means} \qquad e^5 = x$$

Therefore $x = e^5$.

SOLUTION 2 Start with the equation

$$\ln x = 5$$

and apply the exponential function to both sides of the equation:

$$e^{\ln x} = e^5$$

But (4) says that $e^{\ln x} = x$. Therefore $x = e^5$. □

V EXAMPLE 2 Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation and use (5):

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

Since the natural logarithm is found on scientific calculators, we can approxim solution to four decimal places: $x \approx 0.8991$.

The exponential function $f(x) = e^x$ is one of the most frequently occurrin in calculus and its applications, so it is important to be familiar with its graph and its properties (which follow from the fact that it is the inverse of the natura mic function).

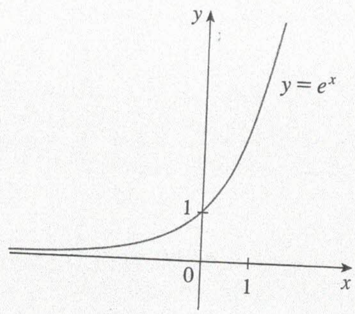


FIGURE 2
The natural exponential function

6 PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION The exponential tion $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x. Also

$$\lim_{x \to -\infty} e^x = 0 \qquad \lim_{x \to \infty} e^x = \infty$$

So the x-axis is a horizontal asymptote of $f(x) = e^x$.

EXAMPLE 3 Find $\displaystyle \lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1}$.

SOLUTION We divide numerator and denominator by e^{2x}:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x} + 1} = \lim_{x \to \infty} \frac{1}{1 + e^{-2x}} = \frac{1}{1 + \lim_{x \to \infty} e^{-2x}}$$

$$= \frac{1}{1 + 0} = 1$$

We have used the fact that $t = -2x \to -\infty$ as $x \to \infty$ and so

$$\lim_{x \to \infty} e^{-2x} = \lim_{t \to -\infty} e^t = 0$$

We now verify that $f(x) = e^x$ has the properties expected of an exponential fu

7 LAWS OF EXPONENTS If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \dfrac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

PROOF OF LAW 1 Using the first law of logarithms and Equation 5, we have

$$\ln(e^x e^y) = \ln(e^x) + \ln(e^y) = x + y = \ln(e^{x+y})$$

Since ln is a one-to-one function, it follows that $e^x e^y = e^{x+y}$.

Laws 2 and 3 are proved similarly (see Exercises 95 and 96). As we will see in t next section, Law 3 actually holds when r is any real number.

TEC Visual 7.2/7.3* uses the slope-a-scope to illustrate this formula.

DIFFERENTIATION

The natural exponential function has the remarkable property that *it is its own derivative.*

8 $$\frac{d}{dx}(e^x) = e^x$$

PROOF The function $y = e^x$ is differentiable because it is the inverse function of $y = \ln x$, which we know is differentiable with nonzero derivative. To find its derivative, we use the inverse function method. Let $y = e^x$. Then $\ln y = x$ and, differentiating this latter equation implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = y = e^x \qquad \square$$

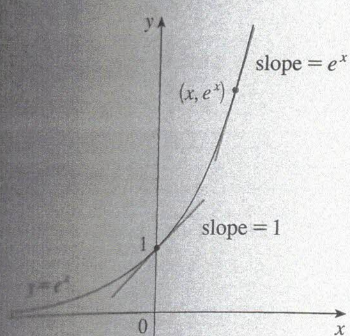


FIGURE 3

The geometric interpretation of Formula 8 is that the slope of a tangent line to the curve $y = e^x$ at any point is equal to the y-coordinate of the point (see Figure 3). This property implies that the exponential curve $y = e^x$ grows very rapidly (see Exercise 100).

V EXAMPLE 4 Differentiate the function $y = e^{\tan x}$.

SOLUTION To use the Chain Rule, we let $u = \tan x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\frac{du}{dx} = e^{\tan x}\sec^2 x \qquad \square$$

In general, if we combine Formula 8 with the Chain Rule, as in Example 4, we get

9 $$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

EXAMPLE 5 Find y' if $y = e^{-4x}\sin 5x$.

SOLUTION Using Formula 9 and the Product Rule, we have

$$y' = e^{-4x}(\cos 5x)(5) + (\sin 5x)e^{-4x}(-4) = e^{-4x}(5\cos 5x - 4\sin 5x) \qquad \square$$

V EXAMPLE 6 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

SOLUTION We differentiate to find any critical numbers:

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, when $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \approx 0.37 \qquad \square$$

EXAMPLE 7 Use the first and second derivatives of $f(x) = e^{1/x}$, together with asy totes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asym by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1$ so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x$ - so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

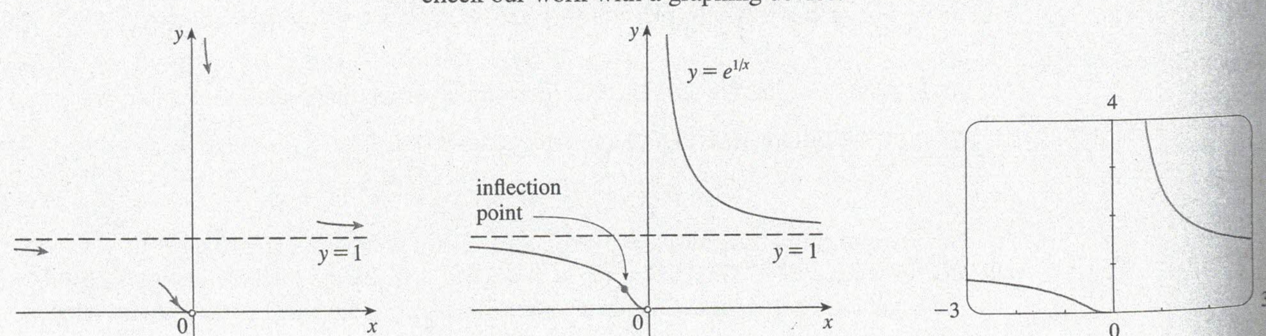
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x)$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upw $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a d line), together with the parts of the curve near the asymptotes in a preliminary sk [Figure 4(a)]. These parts reflect the information concerning limits and the fact t decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x)$ - $x \to 0^-$ even though $f(0)$ does not exist. In Figure 4(b) we finish the sketch by i rating the information concerning concavity and the inflection point. In Figure 4 check our work with a graphing device.



(a) Preliminary sketch

(b) Finished sketch

(c) Computer confirmation

FIGURE 4

INTEGRATION

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

10 $$\int e^x\,dx = e^x + C$$

V EXAMPLE 8 Evaluate $\int x^2 e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$ □

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big]_0^1 = \frac{1}{3}(1 - e^{-3})$$ □

7.3* EXERCISES

1. Sketch, by hand, the graph of the function $f(x) = e^x$ with particular attention to how the graph crosses the y-axis. What fact allows you to do this?

2–4 Simplify each expression.

2. (a) $e^{\ln 15}$ (b) $\ln(1/e)$

3. (a) $e^{-2\ln 5}$ (b) $\ln\left(\ln e^{e^{10}}\right)$

4. (a) $\ln e^{\sin x}$ (b) $e^{x + \ln x}$

5–12 Solve each equation for x.

5. (a) $2\ln x = 1$ (b) $e^{-x} = 5$

6. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

7. (a) $e^{3x+1} = k$ (b) $\ln x + \ln(x - 1) = 1$

8. (a) $\ln(\ln x) = 1$ (b) $e^{e^x} = 10$

9. $3xe^x + x^2e^x = 0$

10. $10(1 + e^{-x})^{-1} = 3$

11. $e^{2x} - e^x - 6 = 0$

12. $\ln(2x + 1) = 2 - \ln x$

13–14 Find the solution of the equation correct to four decimal places.

13. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

14. (a) $\ln\left(1 + \sqrt{x}\right) = 2$ (b) $e^{1/(x-4)} = 7$

15–16 Solve each inequality for x.

15. (a) $e^x < 10$ (b) $\ln x > -1$

16. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

17–20 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graph given in Figure 2 and, if necessary, the transformations of Section 1.3.

17. $y = e^{-x}$

18. $y = 1 + 2e^x$

19. $y = 1 - \frac{1}{2}e^{-x}$

20. $y = 2(1 - e^x)$

21–22 Find (a) the domain of f and (b) f^{-1} and its domain.

21. $f(x) = \sqrt{3 - e^{2x}}$

22. $f(x) = \ln(2 + \ln x)$

23–26 Find the inverse function.

23. $y = \ln(x + 3)$

24. $y = (\ln x)^2, \quad x \geq 1$

25. $f(x) = e^{x^3}$

26. $y = \dfrac{e^x}{1 + 2e^x}$

27–32 Find the limit.

27. $\displaystyle\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

28. $\displaystyle\lim_{x\to\infty} e^{-x^2}$

29. $\displaystyle\lim_{x\to 2^+} e^{3/(2-x)}$

30. $\displaystyle\lim_{x\to 2^-} e^{3/(2-x)}$

31. $\displaystyle\lim_{x\to\infty} (e^{-2x}\cos x)$

32. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\tan x}$

33–48 Differentiate the function.

33. $f(x) = (x^3 + 2x)e^x$

34. $y = \dfrac{e^x}{1 + x}$

35. $y = e^{ax^3}$

36. $y = e^u(\cos u + cu)$

37. $f(u) = e^{1/u}$

38. $g(x) = \sqrt{x}\, e^x$

39. $F(t) = e^{t \sin 2t}$

40. $f(t) = \sin(e^t) + e^{\sin t}$

41. $y = \sqrt{1 + 2e^{3x}}$

42. $y = e^{k \tan \sqrt{x}}$

43. $y = e^{e^x}$

44. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$

45. $y = \dfrac{ae^x + b}{ce^x + d}$

46. $y = \sqrt{1 + xe^{-2x}}$

47. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

48. $f(t) = \sin^2(e^{\sin^2 t})$

49–50 Find an equation of the tangent line to the curve at the given point.

49. $y = e^{2x} \cos \pi x, \quad (0, 1)$

50. $y = e^x/x, \quad (1, e)$

51. Find y' if $e^{x^2 y} = x + y$.

52. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

53. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

54. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

55. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

56. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

57. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

58. Find the thousandth derivative of $f(x) = xe^{-x}$.

59. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

60. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

61. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 10.4 we will see that this is a reasonable equation for $p(t)$.]
(a) Find $\lim_{t \to \infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measure hours. Use the graph to estimate how long it will tak 80% of the population to hear the rumor.

62. For the period from 1980 to 2000, the percentage of hou holds in the United States with at least one VCR has bee modeled by the function

$$V(t) = \frac{85}{1 + 53e^{-0.5t}}$$

where the time t is measured in years since midyear 198 $0 \leqslant t \leqslant 20$. Use a graph to estimate the time at which th number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

63. Find the absolute maximum value of the function $f(x) = x - e^x$.

64. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

65–66 Find the absolute maximum and absolute minimum v of f on the given interval.

65. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

66. $f(x) = x^2e^{-x/2}, \quad [-$

67–68 Find (a) the intervals of increase or decrease, (b) the i vals of concavity, and (c) the points of inflection.

67. $f(x) = (1 - x)e^{-x}$

68. $f(x) = \dfrac{e^x}{x^2}$

69–70 Discuss the curve using the guidelines of Section 4.5.

69. $y = e^{-1/(x+1)}$

70. $y = e^{2x} - e^x$

71. A *drug response curve* describes the level of medication i the bloodstream after a drug is administered. A surge function $S(t) = At^pe^{-kt}$ is often used to model the respons curve, reflecting an initial surge in the drug level and then more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate th times corresponding to the inflection points and explain th significance. If you have a graphing device, use it to grap the drug response curve.

72–73 Draw a graph of f that shows all the important aspect the curve. Estimate the local maximum and minimum values then use calculus to find these values exactly. Use a graph of to estimate the inflection points.

72. $f(x) = e^{\cos x}$

73. $f(x) = e^{x^3 - x}$

74. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

75–84 Evaluate the integral.

75. $\int_0^5 e^{-3x}\, dx$

76. $\int_0^1 xe^{-x^2}\, dx$

77. $\int e^x\sqrt{1 + e^x}\, dx$

78. $\int \frac{(1 + e^x)^2}{e^x}\, dx$

79. $\int (e^x + e^{-x})^2\, dx$

80. $\int e^x(4 + e^x)^5\, dx$

81. $\int \sin x\, e^{\cos x}\, dx$

82. $\int \frac{e^{1/x}}{x^2}\, dx$

83. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

84. $\int e^x \sin(e^x)\, dx$

---

85. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

86. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

87. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

88. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

89. The **error function**

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is used in probability, statistics, and engineering.
(a) Show that $\int_a^b e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}\,[\text{erf}(b) - \text{erf}(a)]$.
(b) Show that the function $y = e^{x^2}\text{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

90. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

91. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

92. Evaluate $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

93. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

94. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

95. Prove the second law of exponents [see (7)].

96. Prove the third law of exponents [see (7)].

97. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\, dx \leqslant e$.

98. (a) Use the inequality of Exercise 97(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\, dx$ given in Exercise 97(b).

99. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

100. This exercise illustrates Exercise 99(c) for the case $k = 10$.
(a) Compare the rates of growth of $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
(b) Find a viewing rectangle that shows how the function $h(x) = e^x/x^{10}$ behaves for large x.
(c) Find a number N such that

$$\text{if } x > N \quad \text{then} \quad \frac{e^x}{x^{10}} > 10^{10}$$

7.4* GENERAL LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we use the natural exponential and logarithmic functions to study exponential and logarithmic functions with base $a > 0$.

GENERAL EXPONENTIAL FUNCTIONS

If $a > 0$ and r is any rational number, then by (4) and (7) in Section 7.3*,

$$a^r = (e^{\ln a})^r = e^{r \ln a}$$

Therefore, even for irrational numbers x, we *define*

1
$$a^x = e^{x \ln a}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = a^x$ is called the **exponential function with base** a. Notice that a^x is positive for all x because e^x is positive for all x.

Definition 1 allows us to extend one of the laws of logarithms. We know that $\ln(a^r) = r \ln a$ when r is rational. But if we now let r be *any* real number we have, from Definition 1,

$$\ln a^r = \ln(e^{r \ln a}) = r \ln a$$

Thus

2
$$\ln a^r = r \ln a \qquad \text{for any real number } r$$

The general laws of exponents follow from Definition 1 together with the laws of exponents for e^x.

3 **LAWS OF EXPONENTS** If x and y are real numbers and $a, b > 0$, then

1. $a^{x+y} = a^x a^y$ **2.** $a^{x-y} = a^x / a^y$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

PROOF

1. Using Definition 1 and the laws of exponents for e^x, we have

$$\begin{aligned} a^{x+y} &= e^{(x+y)\ln a} = e^{x \ln a + y \ln a} \\ &= e^{x \ln a} e^{y \ln a} = a^x a^y \end{aligned}$$

3. Using Equation 2 we obtain

$$\begin{aligned} (a^x)^y &= e^{y \ln(a^x)} = e^{yx \ln a} \\ &= e^{xy \ln a} = a^{xy} \end{aligned}$$

The remaining proofs are left as exercises.

The differentiation formula for exponential functions is also a consequence of Definition 1:

4
$$\frac{d}{dx}(a^x) = a^x \ln a$$

PROOF
$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a}\frac{d}{dx}(x \ln a) = a^x \ln a$$ □

Notice that if $a = e$, then $\ln e = 1$ and Formula 4 simplifies to a formula that we already know: $(d/dx)\, e^x = e^x$. In fact, the reason that the natural exponential function is used more often than other exponential functions is that its differentiation formula is simpler.

EXAMPLE 1 In Example 6 in Section 3.7 we considered a population of bacteria cells in a homogeneous nutrient medium. We showed that if the population doubles every hour, then the population after t hours is

$$n = n_0 2^t$$

where n_0 is the initial population. Now we can use (4) to compute the growth rate:

$$\frac{dn}{dt} = n_0 2^t \ln 2$$

For instance, if the initial population is $n_0 = 1000$ cells, then the growth rate after two hours is

$$\left.\frac{dn}{dt}\right|_{t=2} = (1000)2^t \ln 2\,\Big|_{t=2}$$
$$= 4000 \ln 2 \approx 2773 \text{ cells/h}$$ □

EXAMPLE 2 Combining Formula 4 with the Chain Rule, we have

$$\frac{d}{dx}\left(10^{x^2}\right) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2 \ln 10)x10^{x^2}$$ □

EXPONENTIAL GRAPHS

If $a > 1$, then $\ln a > 0$, so $(d/dx)\, a^x = a^x \ln a > 0$, which shows that $y = a^x$ is increasing (see Figure 1). If $0 < a < 1$, then $\ln a < 0$ and so $y = a^x$ is decreasing (see Figure 2).

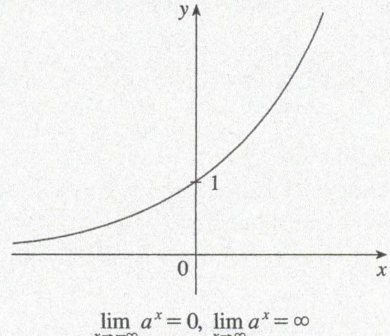


FIGURE 1 $y = a^x,\ a > 1$

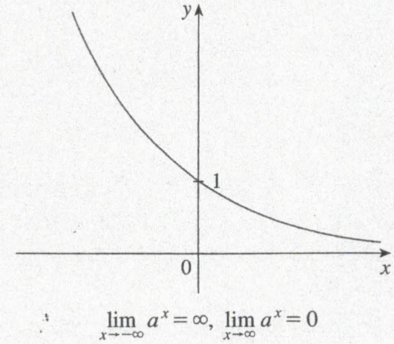


FIGURE 2 $y = a^x,\ 0 < a < 1$

Notice from Figure 3 that as the base a gets larger, the exponential function grows rapidly (for $x > 0$).

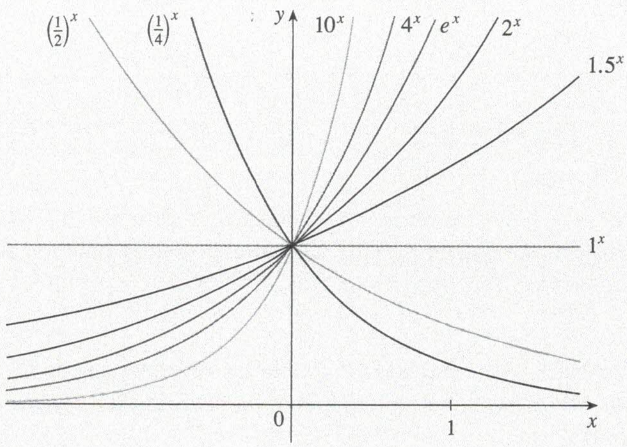


FIGURE 3
Members of the family of exponential functions

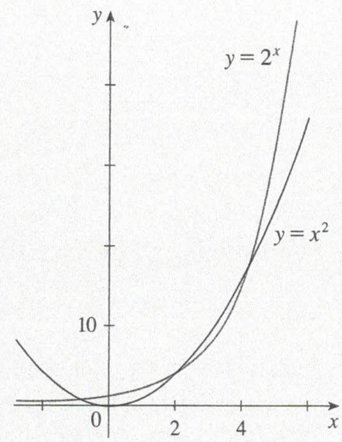


FIGURE 4

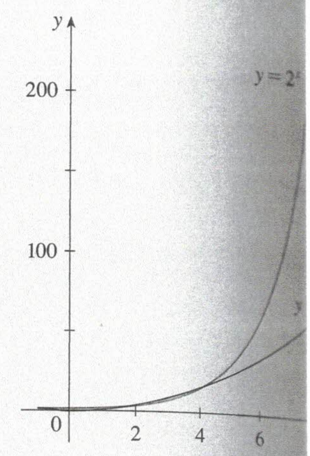


FIGURE 5

Figure 4 shows how the exponential function $y = 2^x$ compares with the power func $y = x^2$. The graphs intersect three times, but ultimately the exponential curve y grows far more rapidly than the parabola $y = x^2$. (See also Figure 5.)

In Section 7.5 we will show how exponential functions occur in the description of ulation growth and radioactive decay. Let's look at human population growth. Tab shows data for the population of the world in the 20th century and Figure 6 shows the responding scatter plot.

TABLE 1

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

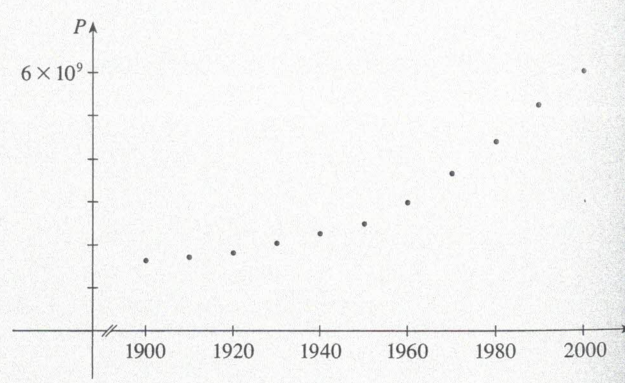


FIGURE 6 Scatter plot for world population growth

The pattern of the data points in Figure 6 suggests exponential growth, so we us graphing calculator with exponential regression capability to apply the method of le squares and obtain the exponential model

$$P = (0.008079266) \cdot (1.013731)^t$$

Figure 7 shows the graph of this exponential function together with the original d points. We see that the exponential curve fits the data reasonably well. The period of r tively slow population growth is explained by the two world wars and the Great Depression of the 1930s.

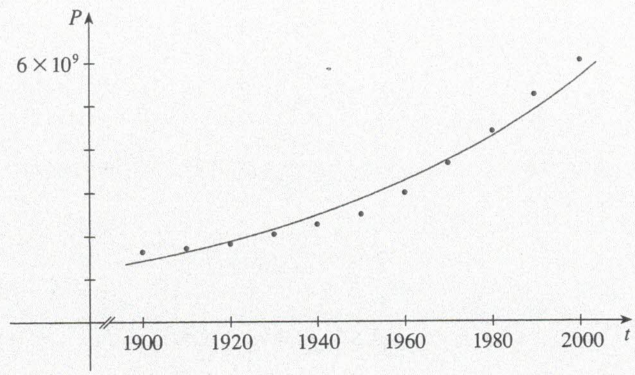


FIGURE 7
Exponential model for population growth

EXPONENTIAL INTEGRALS

The integration formula that follows from Formula 4 is

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad a \neq 1$$

EXAMPLE 3 $\displaystyle \int_0^5 2^x\,dx = \frac{2^x}{\ln 2}\Big]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$ □

THE POWER RULE VERSUS THE EXPONENTIAL RULE

Now that we have defined arbitrary powers of numbers, we are in a position to prove the general version of the Power Rule, as promised in Section 3.3.

THE POWER RULE If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

If $x = 0$, we can show that $f'(0) = 0$ for $n > 1$ directly from the definition of a derivative.

PROOF Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$ □

You should distinguish carefully between the Power Rule $[(d/dx)\,x^n = nx^{n-1}]$, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions $[(d/dx)\,a^x = a^x \ln a]$, where the base is constant and the exponent is variable.

In general there are four cases for exponents and bases:

1. $\dfrac{d}{dx}(a^b) = 0$ (*a* and *b* are constants)

2. $\dfrac{d}{dx}[f(x)]^b = b[f(x)]^{b-1}f'(x)$

3. $\dfrac{d}{dx}[a^{g(x)}] = a^{g(x)}(\ln a)g'(x)$

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 4 Differentiate $y = x^{\sqrt{x}}$.

■ Figure 8 illustrates Example 4 by showing the graphs of $f(x) = x^{\sqrt{x}}$ and its derivative.

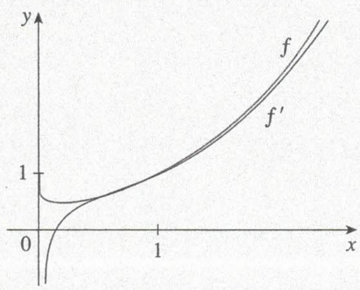


FIGURE 8

SOLUTION 1 Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to write $x^{\sqrt{x}} = (e^{\ln x})^{\sqrt{x}}$:

$$\frac{d}{dx}(x^{\sqrt{x}}) = \frac{d}{dx}(e^{\sqrt{x}\ln x}) = e^{\sqrt{x}\ln x}\frac{d}{dx}(\sqrt{x}\ln x)$$

$$= x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

GENERAL LOGARITHMIC FUNCTIONS

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a one-to-one function. Its inverse function is c the **logarithmic function with base *a*** and is denoted by $\log_a$. Thus

5 $$\log_a x = y \iff a^y = x$$

In particular, we see that

$$\log_e x = \ln x$$

The cancellation equations for the inverse functions $\log_a x$ and a^x are

$$a^{\log_a x} = x \qquad \text{and} \qquad \log_a(a^x) = x$$

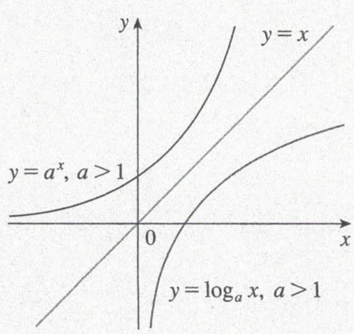


FIGURE 9

Figure 9 shows the case where $a > 1$. (The most important logarithmic functions base $a > 1$.) The fact that $y = a^x$ is a very rapidly increasing function for $x >$ reflected in the fact that $y = \log_a x$ is a very slowly increasing function for $x > 1$.

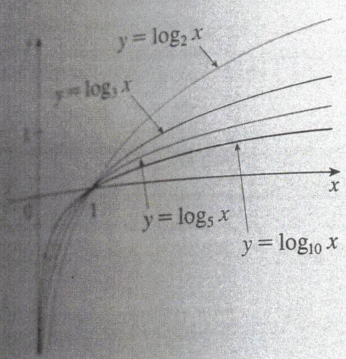


FIGURE 10

Figure 10 shows the graphs of $y = \log_a x$ with various values of the base a. Since $\log_a 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

The laws of logarithms are similar to those for the natural logarithm and can be deduced from the laws of exponents (see Exercise 65).

The following formula shows that logarithms with any base can be expressed in terms of the natural logarithm.

6 CHANGE OF BASE FORMULA For any positive number a $(a \neq 1)$, we have

$$\log_a x = \frac{\ln x}{\ln a}$$

PROOF Let $y = \log_a x$. Then, from (5), we have $a^y = x$. Taking natural logarithms of both sides of this equation, we get $y \ln a = \ln x$. Therefore

$$y = \frac{\ln x}{\ln a}$$ □

Scientific calculators have a key for natural logarithms, so Formula 6 enables us to use a calculator to compute a logarithm with any base (as shown in the following example). Similarly, Formula 6 allows us to graph any logarithmic function on a graphing calculator or computer (see Exercises 14–16).

NOTATION FOR LOGARITHMS
Most textbooks in calculus and the sciences, as well as calculators, use the notation $\ln x$ for the natural logarithm and $\log x$ for the "common logarithm," $\log_{10} x$. In the more advanced mathematical and scientific literature and in computer languages, however, the notation $\log x$ usually denotes the natural logarithm.

EXAMPLE 5 Evaluate $\log_8 5$ correct to six decimal places.

SOLUTION Formula 6 gives

$$\log_8 5 = \frac{\ln 5}{\ln 8} \approx 0.773976$$ □

Formula 6 enables us to differentiate any logarithmic function. Since $\ln a$ is a constant, we can differentiate as follows:

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

7
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

V EXAMPLE 6 Using Formula 7 and the Chain Rule, we get

$$\frac{d}{dx}\log_{10}(2 + \sin x) = \frac{1}{(2 + \sin x)\ln 10}\frac{d}{dx}(2 + \sin x)$$

$$= \frac{\cos x}{(2 + \sin x)\ln 10}$$ □

From Formula 7 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $a = e$ because $\ln e = 1$.

THE NUMBER e AS A LIMIT

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use
to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}$$

8 $$e = \lim_{x \to 0} (1+x)^{1/x}$$

Formula 8 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 11
table of values for small values of x.

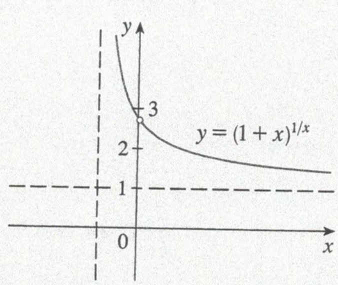


FIGURE 11

x	$(1+x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expre
for e is

9 $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

7.4* EXERCISES

1. (a) Write an equation that defines a^x when a is a positive number and x is a real number.
   (b) What is the domain of the function $f(x) = a^x$?
   (c) If $a \neq 1$, what is the range of this function?
   (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
   (i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) If a is a positive number and $a \neq 1$, how is $\log_a x$ defined?
   (b) What is the domain of the function $f(x) = \log_a x$?
   (c) What is the range of this function?
   (d) If $a > 1$, sketch the general shapes of the graphs of $y = \log_a x$ and $y = a^x$ with a common set of axes.

3–6 Write the expression as a power of e.

3. $5^{\sqrt{7}}$ 4. 10^{x^2}

5. $(\cos x)^x$ 6. $x^{\cos x}$

7–10 Evaluate the expression.

7. (a) $\log_5 125$ (b) $\log_3 \frac{1}{27}$

8. (a) $\log_{10} \sqrt{10}$ (b) $\log_8 320 - \log_8 5$

9. (a) $\log_2 6 - \log_2 15 + \log_2 20$
   (b) $\log_3 100 - \log_3 18 - \log_3 50$

10. (a) $\log_a \dfrac{1}{a}$ (b) $10^{(\log_{10} 4 + \log_{10} 7)}$

11–12 Graph the given functions on a common screen. How are these graphs related?

11. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

12. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

13. Use Formula 6 to evaluate each logarithm correct to six decimal places.
   (a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

14–16 Use Formula 6 to graph the given functions on a common screen. How are these graphs related?

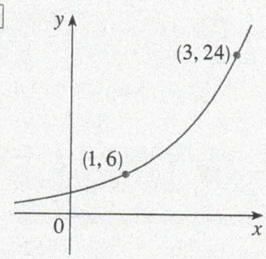
14. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

15. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

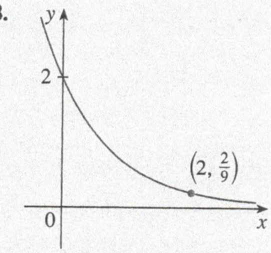
16. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. (a) Show that if the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance 2 ft to the right of the origin the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.
   (b) Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21–24 Find the limit.

21. $\lim_{x \to \infty} (1.001)^x$ 22. $\lim_{x \to -\infty} (1.001)^x$

23. $\lim_{t \to \infty} 2^{-t^2}$ 24. $\lim_{x \to 3^+} \log_{10}(x^2 - 5x + 6)$

25–42 Differentiate the function.

25. $h(t) = t^3 - 3^t$ 26. $g(x) = x^4 4^x$

27. $y = 5^{-1/x}$ 28. $y = 10^{\tan \theta}$

29. $f(u) = (2^u + 2^{-u})^{10}$ 30. $y = 2^{3^{x^2}}$

31. $f(x) = \log_2(1 - 3x)$ 32. $f(x) = \log_5(xe^x)$

33. $y = 2x \log_{10} \sqrt{x}$ 34. $y = \log_2(e^{-x} \cos \pi x)$

35. $y = x^x$ 36. $y = x^{\cos x}$

37. $y = x^{\sin x}$ 38. $y = \sqrt{x}^{\,x}$

39. $y = (\cos x)^x$ 40. $y = (\sin x)^{\ln x}$

41. $y = (\tan x)^{1/x}$ 42. $y = (\ln x)^{\cos x}$

43. Find an equation of the tangent line to the curve $y = 10^x$ at the point $(1, 10)$.

44. If $f(x) = x^{\cos x}$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

45–50 Evaluate the integral.

45. $\int_1^2 10^t \, dt$

46. $\int (x^5 + 5^x) \, dx$

47. $\int \frac{\log_{10} x}{x} \, dx$

48. $\int x 2^{x^2} \, dx$

49. $\int 3^{\sin\theta} \cos\theta \, d\theta$

50. $\int \frac{2^x}{2^x + 1} \, dx$

---

51. Find the area of the region bounded by the curves $y = 2^x$, $y = 5^x$, $x = -1$, and $x = 1$.

52. The region under the curve $y = 10^{-x}$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.

53. Use a graph to find the root of the equation $2^x = 1 + 3^{-x}$ correct to one decimal place. Then use this estimate as the initial approximation in Newton's method to find the root correct to six decimal places.

54. Find y' if $x^y = y^x$.

55. Find the inverse function of $f(x) = \log_{10}\left(1 + \frac{1}{x}\right)$.

56. Calculate $\lim_{x \to \infty} x^{-\ln x}$.

57. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

58. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10 \log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

59. Referring to Exercise 58, find the rate of change of the loudness with respect to the intensity when the sound is measured at 50 dB (the level of ordinary conversation).

60. According to the Beer-Lambert Law, the light intensity at a depth of x meters below the surface of the ocean is $I(x) = I_0 a^x$, where I_0 is the light intensity at the surface and a is a constant such that $0 < a < 1$.

(a) Express the rate of change of $I(x)$ with respect to x in terms of $I(x)$.

(b) If $I_0 = 8$ and $a = 0.38$, find the rate of change of intensity with respect to depth at a depth of 20 m.

(c) Using the values from part (b), find the average light intensity between the surface and a depth of 20 m.

61. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.1
Q	100.00	81.87	67.03	54.88	44.93	36.7

(a) Use a graphing calculator or computer to find an exponential model for the charge.

(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

62. The table gives the US population from 1790 to 1860.

Year	Population	Year	Population
1790	3,929,000	1830	12,861,000
1800	5,308,000	1840	17,063,000
1810	7,240,000	1850	23,192,000
1820	9,639,000	1860	31,443,000

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.

(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).

(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

63. Prove the second law of exponents [see (3)].

64. Prove the fourth law of exponents [see (3)].

65. Deduce the following laws of logarithms from (3):

(a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a(x/y) = \log_a x - \log_a y$

(c) $\log_a(x^y) = y \log_a x$

66. Show that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

7.5 EXPONENTIAL GROWTH AND DECAY

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$; that is, $f'(t) = kf(t)$ for some constant k. Indeed, under ideal conditions (unlimited environment, adequate nutrition, immunity to disease) the mathematical model given by the equation $f'(t) = kf(t)$ predicts what actually happens fairly accurately. Another example occurs in nuclear physics where the mass of a radioactive substance decays at a rate proportional to the mass. In chemistry, the rate of a unimolecular first-order reaction is proportional to the concentration of the substance. In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

In general, if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

1
$$\frac{dy}{dt} = ky$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). It is called a **differential equation** because it involves an unknown function y and its derivative dy/dt.

It's not hard to think of a solution of Equation 1. This equation asks us to find a function whose derivative is a constant multiple of itself. We have met such functions in this chapter. Any exponential function of the form $y(t) = Ce^{kt}$, where C is a constant, satisfies

$$y'(t) = C(ke^{kt}) = k(Ce^{kt}) = ky(t)$$

We will see in Section 10.4 that *any* function that satisfies $dy/dt = ky$ must be of the form $y = Ce^{kt}$. To see the significance of the constant C, we observe that

$$y(0) = Ce^{k \cdot 0} = C$$

Therefore C is the initial value of the function.

2 THEOREM The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}$$

POPULATION GROWTH

What is the significance of the proportionality constant k? In the context of population growth, where $P(t)$ is the size of a population at time t, we can write

3
$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dt} = k$$

The quantity

$$\frac{1}{P}\frac{dP}{dt}$$

is the growth rate divided by the population size; it is called the **relative growth r** According to (3), instead of saying "the growth rate is proportional to population size" could say "the relative growth rate is constant." Then (2) says that a population with stant relative growth rate must grow exponentially. Notice that the relative growth r appears as the coefficient of t in the exponential function Ce^{kt}. For instance, if

$$\frac{dP}{dt} = 0.02P$$

and t is measured in years, then the relative growth rate is $k = 0.02$ and the popula grows at a relative rate of 2% per year. If the population at time 0 is P_0, then the exp sion for the population is

$$P(t) = P_0 e^{0.02t}$$

EXAMPLE 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 2 century. (Assume that the growth rate is proportional to the population size.) What is relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

SOLUTION We measure the time t in years and let $t = 0$ in the year 1950. We measure th population $P(t)$ in millions of people. Then $P(0) = 2560$ and $P(10) = 3040$. Since we are assuming that $dP/dt = kP$, Theorem 2 gives

$$P(t) = P(0)e^{kt} = 2560e^{kt}$$

$$P(10) = 2560e^{10k} = 3040$$

$$k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017185$$

The relative growth rate is about 1.7% per year and the model is

$$P(t) = 2560e^{0.017185t}$$

We estimate that the world population in 1993 was

$$P(43) = 2560e^{0.017185(43)} \approx 5360 \text{ million}$$

The model predicts that the population in 2020 will be

$$P(70) = 2560e^{0.017185(70)} \approx 8524 \text{ million}$$

The graph in Figure 1 shows that the model is fairly accurate to the end of the 20th c tury (the dots represent the actual population), so the estimate for 1993 is quite reliab But the prediction for 2020 is riskier.

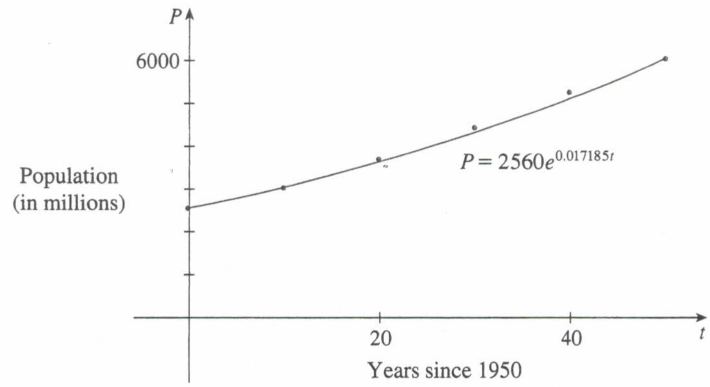


FIGURE 1
A model for world population growth in the second half of the 20th century

RADIOACTIVE DECAY

Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

has been found experimentally to be constant. (Since dm/dt is negative, the relative decay rate is positive.) It follows that

$$\frac{dm}{dt} = km$$

where k is a negative constant. In other words, radioactive substances decay at a rate proportional to the remaining mass. This means that we can use (2) to show that the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Physicists express the rate of decay in terms of **half-life**, the time required for half of any given quantity to decay.

V EXAMPLE 2 The half-life of radium-226 is 1590 years.
(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
(b) Find the mass after 1000 years correct to the nearest milligram.
(c) When will the mass be reduced to 30 mg?

SOLUTION
(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. Then $dm/dt = km$ and $y(0) = 100$, so (2) gives

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

In order to determine the value of k, we use the fact that $y(1590) = \frac{1}{2}(100)$. Thus

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \tfrac{1}{2}$$

and

$$1590k = \ln \tfrac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

We could use the fact that $e^{\ln 2} = 2$ to write the expression for $m(t)$ in the altern form

$$m(t) = 100 \times 2^{-t/1590}$$

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

(c) We want to find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \qquad \text{or} \qquad e^{-(\ln 2)t/1590} = 0.3$$

We solve this equation for t by taking the natural logarithm of both sides:

$$-\frac{\ln 2}{1590} t = \ln 0.3$$

Thus
$$t = -1590 \frac{\ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

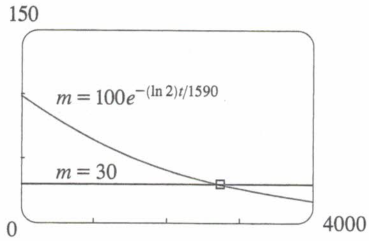


FIGURE 2

As a check on our work in Example 2, we use a graphing device to draw the $m(t)$ in Figure 2 together with the horizontal line $m = 30$. These curves inters $t \approx 2800$, and this agrees with the answer to part (c).

NEWTON'S LAW OF COOLING

Newton's Law of Cooling states that the rate of cooling of an object is propo the temperature difference between the object and its surroundings, provided difference is not too large. (This law also applies to warming.) If we let $T(t)$ be perature of the object at time t and T_s be the temperature of the surroundings, can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so w the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use (2) to find an expression for y, from which we can find T.

EXAMPLE 3 A bottle of soda pop at room temperature (72°F) is placed in a refrige where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F
(a) What is the temperature of the soda pop after another half hour?
(b) How long does it take for the soda pop to cool to 50°F?

SOLUTION
(a) Let $T(t)$ be the temperature of the soda after t minutes. The surrounding temp is $T_s = 44$°F, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \qquad y(0) = 28$$

and by (2) we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

We are given that $T(30) = 61$, so $y(30) = 61 - 44 = 17$ and

$$28e^{30k} = 17 \qquad e^{30k} = \tfrac{17}{28}$$

Taking logarithms, we have

$$k = \frac{\ln\left(\frac{17}{28}\right)}{30} \approx -0.01663$$

Thus

$$y(t) = 28e^{-0.01663t}$$

$$T(t) = 44 + 28e^{-0.01663t}$$

$$T(60) = 44 + 28e^{-0.01663(60)} \approx 54.3$$

So after another half hour the pop has cooled to about 54°F.

(b) We have $T(t) = 50$ when

$$44 + 28e^{-0.01663t} = 50$$

$$e^{-0.01663t} = \tfrac{6}{28}$$

$$t = \frac{\ln\left(\frac{6}{28}\right)}{-0.01663} \approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes. □

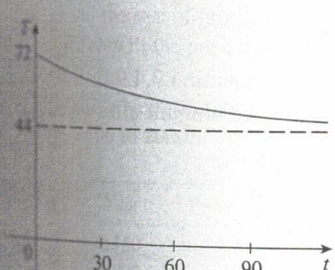


FIGURE 3

Notice that in Example 3, we have

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} (44 + 28e^{-0.01663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. The graph of the temperature function is shown in Figure 3.

CONTINUOUSLY COMPOUNDED INTEREST

EXAMPLE 4 If \$1000 is invested at 6% interest, compounded annually, then after 1 year the investment is worth $\$1000(1.06) = \$1060$, after 2 years it's worth $\$[1000(1.06)]1.06 = \$1123.60$, and after t years it's worth $\$1000(1.06)^t$. In general, if an amount A_0 is invested at an interest rate r ($r = 0.06$ in this example), then after t years it's worth $A_0(1 + r)^t$. Usually, however, interest is compounded more frequently, say, n times a year. Then in each compounding period the interest rate is r/n and there are nt compounding periods in t years, so the value of the investment is

$$A_0\left(1 + \frac{r}{n}\right)^{nt}$$

For instance, after 3 years at 6% interest a \$1000 investment will be worth

$$\$1000(1.06)^3 = \$1191.02 \quad \text{with annual compounding}$$

$$\$1000(1.03)^6 = \$1194.05 \quad \text{with semiannual compoundin}$$

$$\$1000(1.015)^{12} = \$1195.62 \quad \text{with quarterly compounding}$$

$$\$1000(1.005)^{36} = \$1196.68 \quad \text{with monthly compounding}$$

$$\$1000\left(1 + \frac{0.06}{365}\right)^{365 \cdot 3} = \$1197.20 \quad \text{with daily compounding}$$

You can see that the interest paid increases as the number of compounding pe increases. If we let $n \to \infty$, then we will be compounding the interest **continu** the value of the investment will be

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} A_0\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \qquad \text{(where } m = n/r\text{)} \end{aligned}$$

But the limit in this expression is equal to the number e. (See Equation 7.4.9 So with continuous compounding of interest at interest rate r, the amount afte

$$A(t) = A_0 e^{rt}$$

If we differentiate this function, we get

$$\frac{dA}{dt} = rA_0 e^{rt} = rA(t)$$

which says that, with continuous compounding of interest, the rate of increas investment is proportional to its size.

Returning to the example of \$1000 invested for 3 years at 6% interest, we with continuous compounding of interest the value of the investment will be

$$A(3) = \$1000e^{(0.06)3} = \$1197.22$$

Notice how close this is to the amount we calculated for daily compounding, But the amount is easier to compute if we use continuous compounding.

7.5 EXERCISES

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.
   (a) Find the relative growth rate.
   (b) Find an expression for the number of cells after t hours.
   (c) Find the number of cells after 8 hours.
   (d) Find the rate of growth after 8 hours.
   (e) When will the population reach 20,000 cells?

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.
   (a) Find an expression for the number of bacteria after t hours.
   (b) Find the number of bacteria after 3 hours.
   (c) Find the rate of growth after 3 hours.
   (d) When will the population reach 10,000?

4. A bacteria culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.
   (a) Find the initial population.
   (b) Find an expression for the population after t hours.
   (c) Find the number of cells after 5 hours.
   (d) Find the rate of growth after 5 hours.
   (e) When will the population reach 200,000?

5. The table gives estimates of the world population, in millions, from 1750 to 2000:

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

   (a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
   (b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
   (c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

6. The table gives the population of the United States, in millions, for the years 1900–2000.

Year	Population	Year	Population
1900	76	1960	179
1910	92	1970	203
1920	106	1980	227
1930	123	1990	250
1940	131	2000	275
1950	150		

   (a) Use the exponential model and the census figures for 1900 and 1910 to predict the population in 2000. Compare with the actual figure and try to explain the discrepancy.
   (b) Use the exponential model and the census figures for 1980 and 1990 to predict the population in 2000. Compare with the actual population. Then use this model to predict the population in the years 2010 and 2020.
   (c) Graph both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones?

7. Experiments show that if the chemical reaction

$$N_2O_5 \rightarrow 2NO_2 + \tfrac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

   (a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.
   (b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

8. Bismuth-210 has a half-life of 5.0 days.
   (a) A sample originally has a mass of 800 mg. Find a formula for the mass remaining after t days.
   (b) Find the mass remaining after 30 days.
   (c) When is the mass reduced to 1 mg?
   (d) Sketch the graph of the mass function.

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
   (a) Find the mass that remains after t years.
   (b) How much of the sample remains after 100 years?
   (c) After how long will only 1 mg remain?

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
   (a) What is the half-life of tritium-3?
   (b) How long would it take the sample to decay to 20% of its original amount?

11. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

12. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

13. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?

(b) When will the turkey have cooled to 100°F?

14. A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute the thermometer reads 12°C.

(a) What will the reading on the thermometer be after one more minute?

(b) When will the thermometer read 6°C?

15. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.

(a) What is the temperature of the drink after 50 minutes?

(b) When will its temperature be 15°C?

16. (a) A cup of coffee has temperature 95°C and takes utes to cool to 61°C in a room with temperature Show that the temperature of the coffee after t m

$$T(t) = 20 + 75e^{-kt}$$

where $k \approx 0.02$.

(b) What is the average temperature of the coffee du first half hour?

17. The rate of change of atmospheric pressure P with re altitude h is proportional to P, provided that the temp is constant. At 15°C the pressure is 101.3 kPa at sea 87.14 kPa at $h = 1000$ m.

(a) What is the pressure at an altitude of 3000 m?

(b) What is the pressure at the top of Mount McKinl altitude of 6187 m?

18. (a) If \$1000 is borrowed at 8% interest, find the amo due at the end of 3 years if the interest is compou (i) annually, (ii) quarterly, (iii) monthly, (iv) week (v) daily, (vi) hourly, and (vii) continuously.

(b) Suppose \$1000 is borrowed and the interest is co pounded continuously. If $A(t)$ is the amount due t years, where $0 \leqslant t \leqslant 3$, graph $A(t)$ for each of est rates 6%, 8%, and 10% on a common screen.

19. (a) If \$3000 is invested at 5% interest, find the value investment at the end of 5 years if the interest is pounded (i) annually, (ii) semiannually, (iii) mon (iv) weekly, (v) daily, and (vi) continuously.

(b) If $A(t)$ is the amount of the investment at time t f case of continuous compounding, write a differen equation and an initial condition satisfied by $A(t)$.

20. (a) How long will it take an investment to double in the interest rate is 6% compounded continuously

(b) What is the equivalent annual interest rate?

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

In this section we apply the ideas of Section 7.1 to find the derivatives of the s inverse trigonometric functions. We have a slight difficulty in this task: Bec trigonometric functions are not one-to-one, they do not have inverse functions. T culty is overcome by restricting the domains of these functions so that they beco to-one.

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, *is* on (see Figure 2). The inverse function of this restricted sine function f exists and is by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

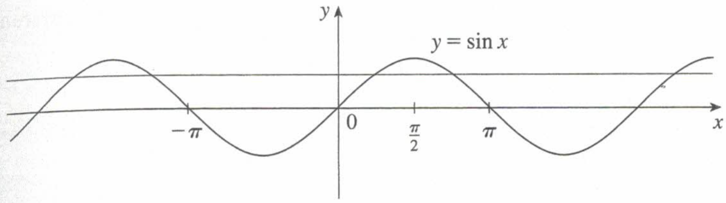


FIGURE 1

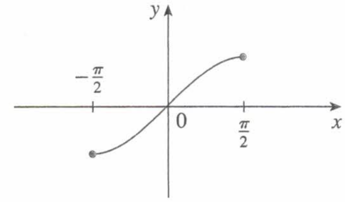


FIGURE 2 $y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\boxed{1} \qquad \boxed{\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

$\sin^{-1}x \neq \dfrac{1}{\sin x}$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

EXAMPLE 1 Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan\left(\arcsin \frac{1}{3}\right)$.

SOLUTION

(a) We have

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ lies between $-\pi/2$ and $\pi/2$.

(b) Let $\theta = \arcsin \frac{1}{3}$, so $\sin \theta = \frac{1}{3}$. Then we can draw a right triangle with angle θ as in Figure 3 and deduce from the Pythagorean Theorem that the third side has length $\sqrt{9 - 1} = 2\sqrt{2}$. This enables us to read from the triangle that

$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan \theta = \frac{1}{2\sqrt{2}}$$ □

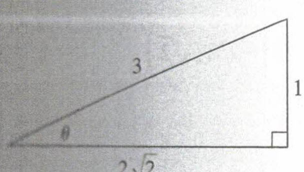


FIGURE 3

The cancellation equations for inverse functions become, in this case,

$$\boxed{2} \qquad \boxed{\begin{aligned} &\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ &\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}}$$

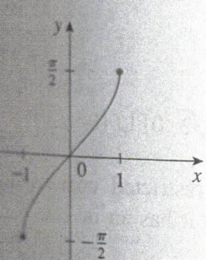


FIGURE 4
$y = \sin^{-1}x = \arcsin x$

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.

We know that the sine function f is continuous, so the inverse sine function is also continuous. We also know from Section 3.4 that the sine function is differentiable, so the inverse sine function is also differentiable. We could calculate the derivative of $\sin^{-1}$ by the formula in Theorem 7.1.7, but since we know that $\sin^{-1}$ is differentiable, we can just as easily calculate it by implicit differentiation as follows.

Let $y = \sin^{-1}x$. Then $\sin y = x$ and $-\pi/2 \leqslant y \leqslant \pi/2$. Differentiating $\sin y = x$ i itly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1$$

and

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$ since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

3

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

V EXAMPLE 2 If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) domain of f'.

SOLUTION

(a) Since the domain of the inverse sine function is $[-1, 1]$, the domain of f is

$$\{x \mid -1 \leqslant x^2 - 1 \leqslant 1\} = \{x \mid 0 \leqslant x^2 \leqslant 2\}$$
$$= \{x \mid |x| \leqslant \sqrt{2}\} = [-\sqrt{2}, \sqrt{2}]$$

(b) Combining Formula 3 with the Chain Rule, we have

$$f'(x) = \frac{1}{\sqrt{1 - (x^2 - 1)^2}} \frac{d}{dx}(x^2 - 1)$$
$$= \frac{1}{\sqrt{1 - (x^4 - 2x^2 + 1)}} 2x = \frac{2x}{\sqrt{2x^2 - x^4}}$$

(c) The domain of f' is

$$\{x \mid -1 < x^2 - 1 < 1\} = \{x \mid 0 < x^2 < 2\}$$
$$= \{x \mid 0 < |x| < \sqrt{2}\} = (-\sqrt{2}, 0) \cup (0, \sqrt{2})$$

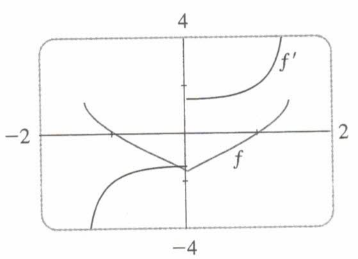


FIGURE 5

The graphs of the function f of Example 2 and its derivative are shown in Figure 5. Notice that f is not differentiable at 0 and this is consistent with the fact that the graph of f' makes a sudden jump at $x = 0$.

The **inverse cosine function** is handled similarly. The restricted cosine f $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 6) and so it has an inverse f denoted by $\cos^{-1}$ or arccos.

4

$$\cos^{-1}x = y \iff \cos y = x \text{ and } 0 \leqslant y \leqslant \pi$$

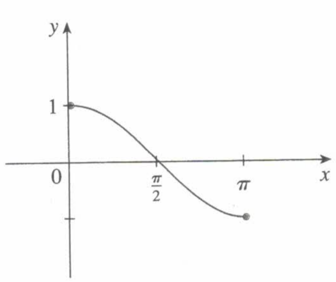


FIGURE 6
$y = \cos x, 0 \leqslant x \leqslant \pi$

The cancellation equations are

5

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$
$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

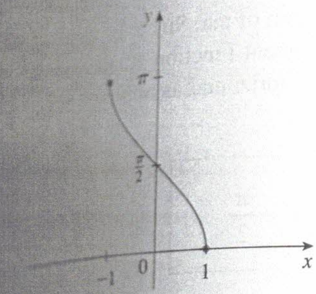


FIGURE 7
$y = \cos^{-1}x = \arccos x$

The inverse cosine function, $\cos^{-1}$, has domain $[-1, 1]$ and range $[0, \pi]$ and is a continuous function whose graph is shown in Figure 7. Its derivative is given by

6
$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

Formula 6 can be proved by the same method as for Formula 3 and is left as Exercise 17.

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$, as shown in Figure 8. It is denoted by $\tan^{-1}$ or arctan.

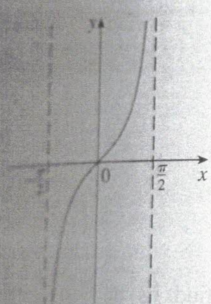


FIGURE 8
$y = \tan x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$

7
$$\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE 3 Simplify the expression $\cos(\tan^{-1}x)$.

SOLUTION 1 Let $y = \tan^{-1}x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\sec y = \sqrt{1 + x^2} \qquad (\text{since } \sec y > 0 \text{ for } -\pi/2 < y < \pi/2)$$

Thus
$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + x^2}}$$

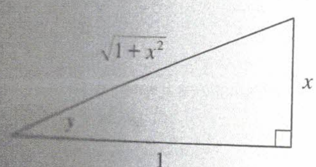


FIGURE 9

SOLUTION 2 Instead of using trigonometric identities as in Solution 1, it is perhaps easier to use a diagram. If $y = \tan^{-1}x$, then $\tan y = x$, and we can read from Figure 9 (which illustrates the case $y > 0$) that

$$\cos(\tan^{-1}x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$ □

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 10.

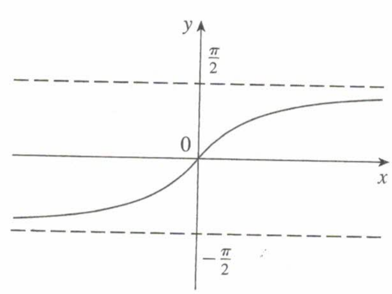


FIGURE 10
$y = \tan^{-1}x = \arctan x$

We know that

$$\lim_{x\to(\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x\to-(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan. Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$. This fact is expressed by the following limits:

8
$$\lim_{x\to\infty} \tan^{-1}x = \frac{\pi}{2} \qquad \lim_{x\to-\infty} \tan^{-1}x = -\frac{\pi}{2}$$

EXAMPLE 4 Evaluate $\displaystyle\lim_{x\to2^+} \arctan\left(\frac{1}{x-2}\right)$.

SOLUTION Since

$$\frac{1}{x-2} \to \infty \qquad \text{as } x \to 2^+$$

the first equation in (8) gives

$$\lim_{x\to2^+} \arctan\left(\frac{1}{x-2}\right) = \frac{\pi}{2}$$

Since tan is differentiable, $\tan^{-1}$ is also differentiable. To find its derivative, let $y = \tan^{-1}x$. Then $\tan y = x$. Differentiating this latter equation implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

and so
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

9
$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

10
$$y = \csc^{-1}x \ (|x| \geqslant 1) \iff \csc y = x \ \text{ and } \ y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$
$$y = \sec^{-1}x \ (|x| \geqslant 1) \iff \sec y = x \ \text{ and } \ y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$
$$y = \cot^{-1}x \ (x \in \mathbb{R}) \iff \cot y = x \ \text{ and } \ y \in (0, \pi)$$

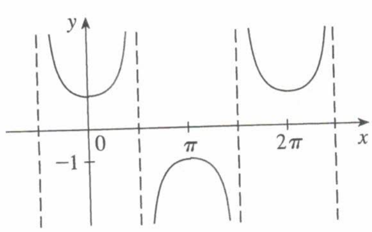


FIGURE 11
$y = \sec x$

The choice of intervals for y in the definitions of $\csc^{-1}$ and $\sec^{-1}$ is not universally agreed upon. For instance, some authors use $y \in [0, \pi/2) \cup (\pi/2, \pi]$ in the definition of $\sec^{-1}$. [You can see from the graph of the secant function in Figure 11 that both this choice and the one in (10) will work.] The reason for the choice in (10) is that the differentiation formulas are simpler (see Exercise 79).

We collect in Table 11 the differentiation formulas for all of the inverse trigonometric functions. The proofs of the formulas for the derivatives of $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ are left as Exercises 19–21.

11 TABLE OF DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Each of these formulas can be combined with the Chain Rule. For instance, if u is a differentiable function of x, then

$$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx}$$

V EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x\arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$
$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

Recall that arctan x is an alternative notation for $\tan^{-1}x$.

(b)
$$f'(x) = x\frac{1}{1+(\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$
$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$
□

EXAMPLE 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \pi/2$.

SOLUTION Although calculus is not needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1}x + \cot^{-1}x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$. Then

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Thus $\tan^{-1}x + \cot^{-1}x = \pi/2$. □

Each of the formulas in Table 11 gives rise to an integration formula. The two most useful of these are the following:

12
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

13
$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

EXAMPLE 7 Find $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx$.

SOLUTION If we write

$$\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \int_0^{1/4} \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

then the integral resembles Equation 12 and the substitution $u = 2x$ is suggested. This gives $du = 2\,dx$, so $dx = du/2$. When $x = 0$, $u = 0$; when $x = \frac{1}{4}$, $u = \frac{1}{2}$. So

$$\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \tfrac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u\Big]_0^{1/2}$$

$$= \tfrac{1}{2}\left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}0\right] = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$$

EXAMPLE 8 Evaluate $\displaystyle\int \frac{1}{x^2+a^2}\,dx$.

SOLUTION To make the given integral more like Equation 13 we write

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left(\dfrac{x^2}{a^2}+1\right)} = \frac{1}{a^2}\int \frac{dx}{\left(\dfrac{x}{a}\right)^2+1}$$

This suggests that we substitute $u = x/a$. Then $du = dx/a$, $dx = a\,du$, and

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\tan^{-1}u + C$$

Thus we have the formula

One of the main uses of inverse trigonometric functions is that they often arise when we integrate rational functions.

14
$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 9 Find $\displaystyle\int \frac{x}{x^4+9}\,dx$.

SOLUTION We substitute $u = x^2$ because then $du = 2x\,dx$ and we can use Equation 14 with $a = 3$:

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

7.6 EXERCISES

1–10 Find the exact value of each expression.

1. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$
2. (a) $\tan^{-1}(1/\sqrt{3})$ (b) $\sec^{-1} 2$
3. (a) $\arctan 1$ (b) $\sin^{-1}(1/\sqrt{2})$
4. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\arccos(-\frac{1}{2})$
5. (a) $\tan(\arctan 10)$ (b) $\sin^{-1}(\sin(7\pi/3))$
6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos(\arcsin \frac{1}{2})$
7. $\tan(\sin^{-1}(\frac{2}{3}))$
8. $\csc(\arccos \frac{3}{5})$
9. $\sin(2\tan^{-1}\sqrt{2})$
10. $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

11. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

12–14 Simplify the expression.

12. $\tan(\sin^{-1}x)$
13. $\sin(\tan^{-1}x)$
14. $\cos(2\tan^{-1}x)$

15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$
16. $y = \tan x,\ -\pi/2 < x < \pi/2;\quad y = \tan^{-1}x;\quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$.

20. Prove that $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$.

21. Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$.

22–35 Find the derivative of the function. Simplify where possible.

22. $y = \sqrt{\tan^{-1}x}$
23. $y = \tan^{-1}\sqrt{x}$
24. $f(x) = x\ln(\arctan x)$
25. $y = \sin^{-1}(2x + 1)$
26. $g(x) = \sqrt{x^2 - 1}\,\sec^{-1}x$
27. $G(x) = \sqrt{1 - x^2}\,\arccos x$
28. $F(\theta) = \arcsin\sqrt{\sin\theta}$
29. $y = \cos^{-1}(e^{2x})$
30. $y = \arctan\sqrt{\dfrac{1 - x}{1 + x}}$
31. $y = \arctan(\cos\theta)$
32. $y = \tan^{-1}(x - \sqrt{1 + x^2})$
33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$
34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$
35. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right),\quad 0 \leqslant x \leqslant \pi,\ a > b > 0$

36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$
37. $g(x) = \cos^{-1}(3 - 2x)$

38. Find y' if $\tan^{-1}(xy) = 1 + x^2y$.

39. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16 - x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

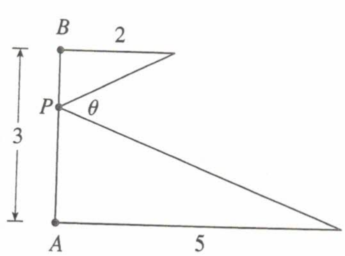
41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = \sqrt{1 - x^2}\arcsin x$
42. $f(x) = \arctan(x^2 - x)$

43–46 Find the limit.

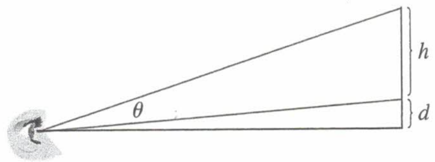
43. $\lim_{x \to -1^+} \sin^{-1}x$
44. $\lim_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$
45. $\lim_{x \to \infty} \arctan(e^x)$
46. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure on page 462). How far from the wall should the observer stand to get the best view? (In other words, where

should the observer stand so as to maximize the angle θ subtended at her eye by the painting?)



49. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 ft/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 Sketch the curve using the guidelines of Section 4.5.

51. $y = \sin^{-1}\left(\frac{x}{x+1}\right)$

52. $y = \tan^{-1}\left(\frac{x-1}{x+1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = \tan^{-1}(\ln x)$

CAS **55.** If $f(x) = \arctan(\cos(3 \arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c\sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.

57. Find the most general antiderivative of the function

$$f(x) = \frac{2 + x^2}{1 + x^2}$$

58. Find $f(x)$ if $f'(x) = 4/\sqrt{1 - x^2}$ and $f\left(\frac{1}{2}\right) = 1$.

59–70 Evaluate the integral.

59. $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1 - t^2}}\,dt$

60. $\int \frac{\tan^{-1}x}{1 + x^2}\,dx$

61. $\int_0^{\sqrt{3}/4} \frac{dx}{1 + 16x^2}$

62. $\int \frac{dt}{\sqrt{1 - 4t^2}}$

63. $\int \frac{1 + x}{1 + x^2}\,dx$

64. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\,dx$

65. $\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1}x}$

66. $\int \frac{1}{x\sqrt{x^2 - 4}}\,dx$

67. $\int \frac{t^2}{\sqrt{1 - t^6}}\,dt$

68. $\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\,dx$

69. $\int \frac{dx}{\sqrt{x}\,(1 + x)}$

70. $\int \frac{x}{1 + x^4}\,dx$

71. Use the method of Example 8 to show that, if $a > 0$,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

72. The region under the curve $y = 1/\sqrt{x^2 + 4}$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume of the resulting solid.

73. Evaluate $\int_0^1 \sin^{-1}x\,dx$ by interpreting it as an area and integrating with respect to y instead of x.

74. Prove that, for $xy \neq 1$,

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

75. Use the result of Exercise 74 to prove the following:
(a) $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \pi/4$
(b) $2 \arctan \frac{1}{3} + \arctan \frac{1}{7} = \pi/4$

76. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}x)$.
(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.
(c) Show that $g'(x) = \frac{\cos x}{|\cos x|}$.
(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

77. Use the method of Example 6 to prove the identity

$$2\sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geqslant 0$$

78. Prove the identity

$$\arcsin \frac{x - 1}{x + 1} = 2 \arctan \sqrt{x} - \frac{\pi}{2}$$

79. Some authors define $y = \sec^{-1}x \iff \sec y = x$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$. Show that with this definition we have (instead of the formula given in Exercise 20)

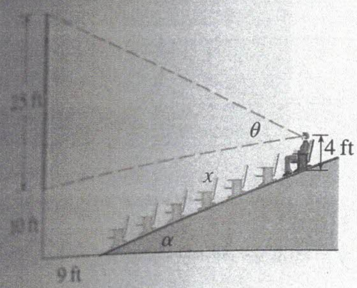
$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

80. Let $f(x) = x\arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.
(a) Is f continuous at 0?
(b) Is f differentiable at 0?

APPLIED PROJECT

CAS WHERE TO SIT AT THE MOVIES

A movie theater has a screen that is positioned 10 ft off the floor and is 25 ft high. The first row of seats is placed 9 ft from the screen and the rows are set 3 ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leqslant x \leqslant 60$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4 ft above the floor, as shown in the figure. (In Exercise 48 in Section 7.6 we looked at a simpler version of this problem, where the floor is horizontal, but this project involves a more complicated situation and requires technology.)



1. Show that

$$\theta = \arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where $$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$

and $$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

2. Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row?

3. Use your computer algebra system to differentiate θ and find a numerical value for the root of the equation $d\theta/dx = 0$. Does this value confirm your result in Problem 2?

4. Use the graph of θ to estimate the average value of θ on the interval $0 \leqslant x \leqslant 60$. Then use your CAS to compute the average value. Compare with the maximum and minimum values of θ.

7.7 HYPERBOLIC FUNCTIONS

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine, hyperbolic cosine**, and so on.

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical ad(in Figures 1 and 2.

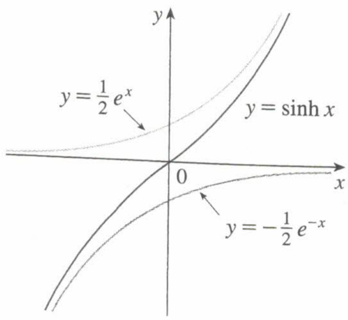


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

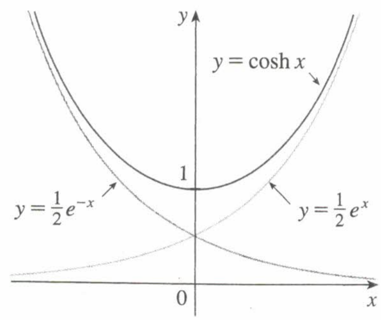


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

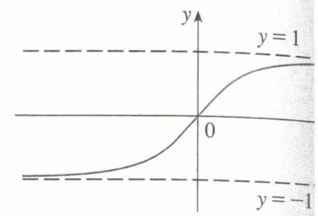


FIGURE 3
$y = \tanh x$

Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, while cosh has domain $\mathbb{R}$ and range The graph of tanh is shown in Figure 3. It has the horizontal asymptotes $y = \pm$ Exercise 23.)

Some of the mathematical uses of hyperbolic functions will be seen in Ch Applications to science and engineering occur whenever an entity such as light, v electricity, or radioactivity is gradually absorbed or extinguished, for the decay can resented by hyperbolic functions. The most famous application is the use of hyp cosine to describe the shape of a hanging wire. It can be proved that if a heavy cable (such as a telephone or power line) is suspended between two points at th height, then it takes the shape of a curve with equation $y = c + a\cosh(x/a)$ called *nary* (see Figure 4). (The Latin word *catena* means "chain.")

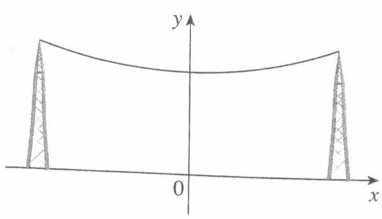


FIGURE 4
A catenary $y = c + a\cosh(x/a)$

Another application of hyperbolic functions occurs in the description of ocean The velocity of a water wave with length L moving across a body of water with de modeled by the function

$$v = \sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5 and Exercise 49.)

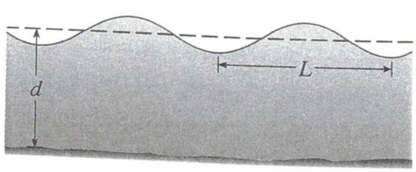


FIGURE 5
Idealized ocean wave

The hyperbolic functions satisfy a number of identities that are similar to well- trigonometric identities. We list some of them here and leave most of the proofs exercises.

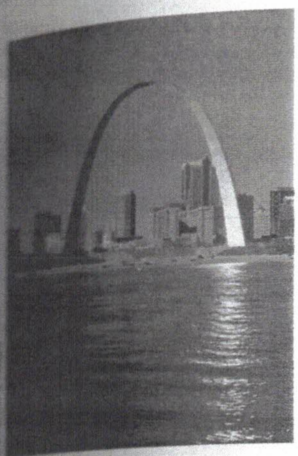
HYPERBOLIC IDENTITIES

$\sinh(-x) = -\sinh x$ $\quad\quad$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $\quad\quad$ $1 - \tanh^2 x = \text{sech}^2 x$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$



The Gateway Arch in St. Louis was designed using a hyperbolic cosine function (Exercise 48).

EXAMPLE 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

SOLUTION

(a)
$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

(b) We start with the identity proved in part (a):

$$\cosh^2 x - \sinh^2 x = 1$$

If we divide both sides by $\cosh^2 x$, we get

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

or

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$ □

The identity proved in Example 1(a) gives a clue to the reason for the name "hyperbolic" functions:

If t is any real number, then the point $P(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ because $\cos^2 t + \sin^2 t = 1$. In fact, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions.

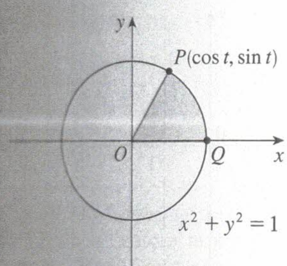


FIGURE 6

Likewise, if t is any real number, then the point $P(\cosh t, \sinh t)$ lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sinh^2 t = 1$ and $\cosh t \geqslant 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular sector in Figure 6.

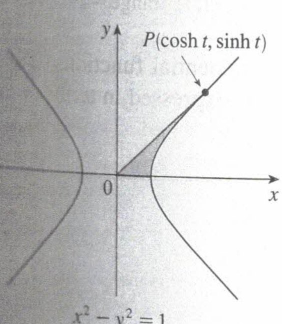


FIGURE 7

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as Table 1. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric functions, but beware that the signs are different in some cases.

1 DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

EXAMPLE 2 Any of these differentiation rules can be combined with the Chain Rule. For instance,

$$\frac{d}{dx}(\cosh \sqrt{x}) = \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

INVERSE HYPERBOLIC FUNCTIONS

You can see from Figures 1 and 3 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Figure 2 shows that cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \text{ and } y \geq 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

The remaining inverse hyperbolic functions are defined similarly (see Exercise 28).

We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10 by using Figures 1, 2, and 3.

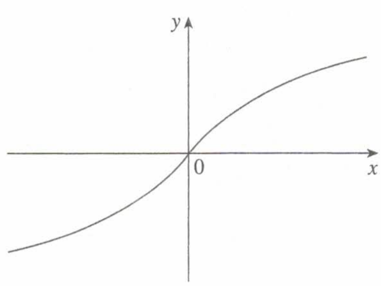


FIGURE 8 $y = \sinh^{-1}x$
domain = $\mathbb{R}$ range = $\mathbb{R}$

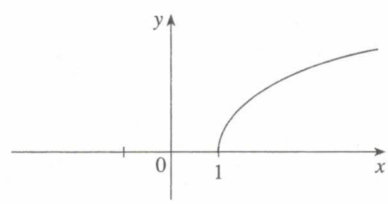


FIGURE 9 $y = \cosh^{-1}x$
domain = $[1, \infty)$ range = $[0, \infty)$

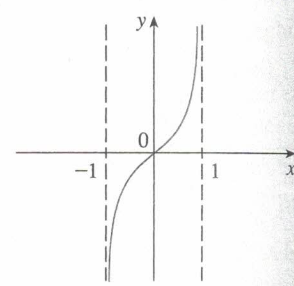


FIGURE 10 $y = \tanh^{-1}x$
domain = $(-1, 1)$ range = $\mathbb{R}$

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

Formula 3 is proved in Example 3. The proofs of Formulas 4 and 5 are requested in Exercises 26 and 27.

3 $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

4 $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1$$

5 $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

EXAMPLE 3 Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

SOLUTION Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

so
$$e^y - 2x - e^{-y} = 0$$

or, multiplying by e^y,
$$e^{2y} - 2xe^y - 1 = 0$$

This is really a quadratic equation in e^y:
$$(e^y)^2 - 2x(e^y) - 1 = 0$$

Solving by the quadratic formula, we get
$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus the minus sign is inadmissible and we have
$$e^y = x + \sqrt{x^2 + 1}$$

Therefore
$$y = \ln(e^y) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

(See Exercise 25 for another method.) □

6 **DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS**

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

Notice that the formulas for the derivatives of $\tanh^{-1}x$ and $\coth^{-1}x$ appear to be identical. But the domains of these functions have no numbers in common: $\tanh^{-1}x$ is defined for $|x| < 1$, whereas $\coth^{-1}x$ is defined for $|x| > 1$.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in Table 6 can be proved either by the method for inverse functions or by differentiating Formulas 3, 4, and 5.

V EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$. If we differentiate this equation implicitly with respect to x, we get
$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$, we have $\cosh y = \sqrt{1 + \sinh^2 y}$, so
$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\begin{aligned}\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}\\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

V EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\begin{aligned}\frac{d}{dx}[\tanh^{-1}(\sin x)] &= \frac{1}{1-(\sin x)^2}\frac{d}{dx}(\sin x)\\ &= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x\end{aligned}$$

V EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$.

SOLUTION Using Table 6 (or Example 4) we know that an antiderivative of $1/\sqrt{1+x^2}$ is $\sinh^{-1}x$. Therefore

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1}x\Big]_0^1\\ &= \sinh^{-1}1\\ &= \ln\left(1+\sqrt{2}\right) \qquad \text{(from Equation 3)}\end{aligned}$$

7.7 EXERCISES

1–6 Find the numerical value of each expression.

1. (a) sinh 0 (b) cosh 0
2. (a) tanh 0 (b) tanh 1
3. (a) sinh(ln 2) (b) sinh 2
4. (a) cosh 3 (b) cosh(ln 3)
5. (a) sech 0 (b) $\cosh^{-1} 1$
6. (a) sinh 1 (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
   (This shows that sinh is an odd function.)
8. $\cosh(-x) = \cosh x$
   (This shows that cosh is an even function.)
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
12. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ (n any real number)

---

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$
(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$
(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$
(g) $\lim_{x\to 0^+} \coth x$ (h) $\lim_{x\to 0^-} \coth x$
(i) $\lim_{x\to-\infty} \operatorname{csch} x$

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

28. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–47 Find the derivative. Simplify where possible.

30. $f(x) = \tanh(1 + e^{2x})$

31. $f(x) = x \sinh x - \cosh x$

32. $g(x) = \cosh(\ln x)$

33. $h(x) = \ln(\cosh x)$

34. $y = x \coth(1 + x^2)$

35. $y = e^{\cosh 3x}$

36. $f(t) = \operatorname{csch} t\,(1 - \ln \operatorname{csch} t)$

37. $f(t) = \operatorname{sech}^2(e^t)$

38. $y = \sinh(\cosh x)$

39. $y = \arctan(\tanh x)$

40. $y = \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}}$

41. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x}$

42. $y = x^2 \sinh^{-1}(2x)$

43. $y = \tanh^{-1}\sqrt{x}$

44. $y = x \tanh^{-1}x + \ln\sqrt{1 - x^2}$

45. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

46. $y = \operatorname{sech}^{-1}\sqrt{1 - x^2}, \quad x > 0$

47. $y = \coth^{-1}\sqrt{x^2 + 1}$

---

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

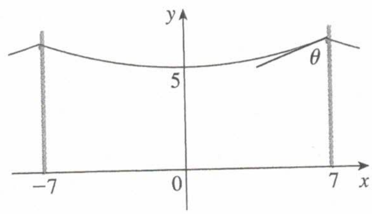
$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

50. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters. (See the diagram on page 470.)
(a) Find the slope of this curve where it meets the right pole.

(b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, and T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. (a) Show that any function of the form

$$y = A\sinh mx + B\cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

54. Evaluate $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{e^x}$.

55. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

56. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

57–65 Evaluate the integral.

57. $\displaystyle\int \sinh x \cosh^2 x\,dx$

58. $\displaystyle\int \sinh(1 + 4x)\,dx$

59. $\displaystyle\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

60. $\displaystyle\int \tanh x\,dx$

61. $\displaystyle\int \frac{\cosh x}{\cosh^2 x - 1}\,dx$

62. $\displaystyle\int \frac{\operatorname{sech}^2 x}{2 + \tanh x}\,dx$

63. $\displaystyle\int_4^6 \frac{1}{\sqrt{t^2 - 9}}\,dt$

64. $\displaystyle\int_0^1 \frac{1}{\sqrt{16t^2 + 1}}\,dt$

65. $\displaystyle\int \frac{e^x}{1 - e^{2x}}\,dx$

66. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

67. (a) Use Newton's method or a graphing device to find approximate solutions of the equation $\cosh 2x = 1 + \sinh x$.

(b) Estimate the area of the region bounded by the curves $y = \cosh 2x$ and $y = 1 + \sinh x$.

68. Show that the area of the shaded hyperbolic sector in Figure 7 is $A(t) = \frac{1}{2}t$. [*Hint:* First show that

$$A(t) = \tfrac{1}{2}\sinh t\cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$$

and then verify that $A'(t) = \frac{1}{2}$.]

69. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha\sinh(x + \beta)$ or $\alpha\cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

7.8 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x\to 1} \frac{\ln x}{x - 1} \qquad \boxed{1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 2.3) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**. We met some limits of this type in Chapter 2. For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

But these methods do not work for limits such as (1), so in this section we introduce a systematic method, known as *l'Hospital's Rule,* for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate its limit at infinity:

$$\boxed{2} \qquad \lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$. There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer may be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞**. We saw in Section 4.4 that this type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs in the denominator. For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as (2), but l'Hospital's Rule also applies to this type of indeterminate form.

L'HOSPITAL

L'Hospital's Rule is named after a French nobleman, the Marquis de l'Hospital (1661–1704), but was discovered by a Swiss mathematician, John Bernoulli (1667–1748). You might sometimes see l'Hospital spelled as l'Hôpital, but he spelled his own name l'Hospital, as was common in the 17th century. See Exercise 80 for the example that the Marquis used to illustrate his rule. See the project on page 481 for further historical details.

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an op interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

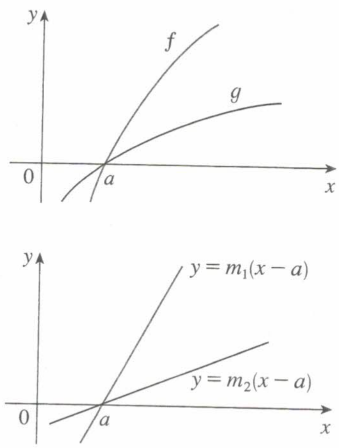


FIGURE 1

Figure 1 suggests visually why l'Hospital's Rule might be true. The first graph shows two differentiable functions f and g, each of which approaches 0 as $x \to a$. If we were to zoom in toward the point $(a, 0)$, the graphs would start to look almost linear. But if the functions actually *were* linear, as in the second graph, then their ratio would be

$$\frac{m_1(x - a)}{m_2(x - a)} = \frac{m_1}{m_2}$$

which is the ratio of their derivatives. This suggests that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equ limit of the quotient of their derivatives, provided that the given conditions are sat is especially important to verify the conditions regarding the limits of f and g befo l'Hospital's Rule.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at in negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, . $x \to \infty$, or $x \to -\infty$.

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continu $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternati of the definition of a derivative, we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}} = \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

The general version of l'Hospital's Rule for the indeterminate form $\frac{0}{0}$ is somewha difficult and its proof is deferred to the end of this section. The proof for the indeter form ∞/∞ can be found in more advanced books.

EXAMPLE 1 Find $\displaystyle\lim_{x \to 1} \frac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \qquad \text{and} \qquad \lim_{x \to 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Notice that when using l'Hospital's Rule we differentiate the numerator and denominator *separately*. We do *not* use the Quotient Rule.

The graph of the function of Example 2 is shown in Figure 2. We have noticed previously that exponential functions grow far more rapidly than power functions, so the result of Example 2 is not unexpected. See also Exercise 93.

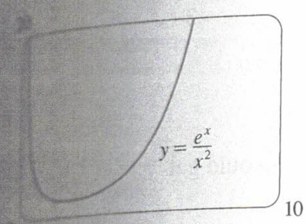


FIGURE 2

EXAMPLE 2 Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ □

The graph of the function of Example 3 is shown in Figure 3. We have discussed previously the slow growth of logarithms, so it isn't surprising that this ratio approaches 0 as $x \to \infty$. See also Exercise 94.

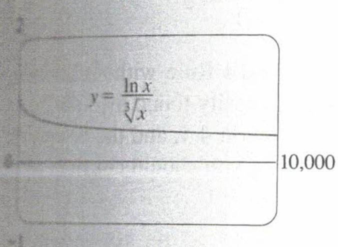


FIGURE 3

V EXAMPLE 3 Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$ □

EXAMPLE 4 Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$. (See Exercise 38 in Section 2.2.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Because $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits.

The graph in Figure 4 gives visual confirmation of the result of Example 4. If we were to zoom in too far, however, we would get an inaccurate graph because $\tan x$ is close to x when x is small. See Exercise 38(d) in Section 2.2.

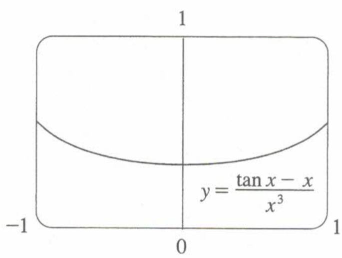


FIGURE 4

Putting together all the steps, we get

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{1}{3}$$

EXAMPLE 5 Find $\displaystyle\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-}\frac{\cos x}{\sin x} = -\infty$$

This is wrong! Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the deno nator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at the denominator is nonzero there:

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = 0$$

Example 5 shows what can go wrong if you use l'Hospital's Rule without th Other limits *can* be found using l'Hospital's Rule but are more easily found by other ods. (See Examples 3 and 5 in Section 2.3, Example 3 in Section 4.4, and the disc at the beginning of this section.) So when evaluating any limit, you should conside methods before using l'Hospital's Rule.

INDETERMINATE PRODUCTS

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the va $\lim_{x\to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise the answer is a finite nonzero number. This kind of limit is called an **indeterminate of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that w use l'Hospital's Rule.

Figure 5 shows the graph of the function in Example 6. Notice that the function is undefined at $x = 0$; the graph approaches the origin but never quite reaches it.

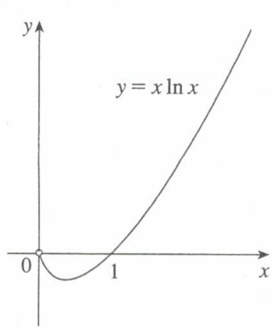


FIGURE 5

EXAMPLE 6 Evaluate $\displaystyle\lim_{x\to 0^+} x\ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, w have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type 0/0, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

EXAMPLE 7 Use l'Hospital's Rule to help sketch the graph of $f(x) = xe^x$.

SOLUTION Because both x and e^x become large as $x \to \infty$, we have $\lim_{x \to \infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} (-e^x) = 0$$

Thus the x-axis is a horizontal asymptote.

We use the methods of Chapter 4 to gather other information concerning the graph. The derivative is

$$f'(x) = xe^x + e^x = (x + 1)e^x$$

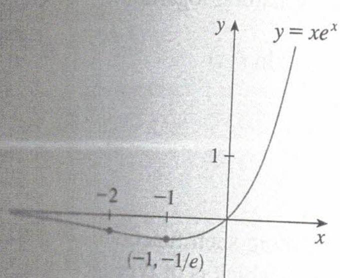


FIGURE 6

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$. Because $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum. The second derivative is

$$f''(x) = (x + 1)e^x + e^x = (x + 2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.

We use this information to sketch the curve in Figure 6. □

INDETERMINATE DIFFERENCES

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

EXAMPLE 8 Compute $\displaystyle\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x) = \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x} = \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

INDETERMINATE POWERS

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x)\ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1+\sin 4x)^{\cot x}$$

Then

$$\ln y = \ln[(1+\sin 4x)^{\cot x}] = \cot x \ln(1+\sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\dfrac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^4$$

The graph of the function $y = x^x$, $x > 0$, is shown in Figure 7. Notice that although 0^0 is not defined, the values of the function approach 1 as $x \to 0^+$. This confirms the result of Example 10.

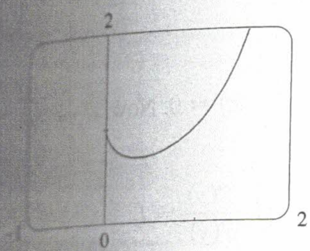


FIGURE 7

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 9 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

□

In order to give the promised proof of l'Hospital's Rule, we first need a generalization of the Mean Value Theorem. The following theorem is named after another French mathematician, Augustin-Louis Cauchy (1789–1857).

See the biographical sketch of Cauchy on page 91.

3 CAUCHY'S MEAN VALUE THEOREM Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Notice that if we take the special case in which $g(x) = x$, then $g'(c) = 1$ and Theorem 3 is just the ordinary Mean Value Theorem. Furthermore, Theorem 3 can be proved in a similar manner. You can verify that all we have to do is change the function h given by Equation 4.2.4 to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

and apply Rolle's Theorem as before.

PROOF OF L'HOSPITAL'S RULE We are assuming that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

We must show that $\lim_{x \to a} f(x)/g(x) = L$. Define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Then F is continuous on I since f is continuous on $\{x \in I \mid x \neq a\}$ and

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = 0 = F(a)$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous o $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Ther fore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we le $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)}$$

$$= \lim_{t \to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)}$$

$$= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

7.8 EXERCISES

1–4 Given that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 1$$

$$\lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$ (b) $\displaystyle\lim_{x \to a} \frac{f(x)}{p(x)}$

(c) $\displaystyle\lim_{x \to a} \frac{h(x)}{p(x)}$ (d) $\displaystyle\lim_{x \to a} \frac{p(x)}{f(x)}$

(e) $\displaystyle\lim_{x \to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x \to a} [f(x)p(x)]$ (b) $\lim_{x \to a} [h(x)p(x)]$

(c) $\lim_{x \to a} [p(x)q(x)]$

3. (a) $\lim_{x \to a} [f(x) - p(x)]$ (b) $\lim_{x \to a} [p(x) - q(x)]$

(c) $\lim_{x \to a} [p(x) + q(x)]$

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ (b) $\lim_{x \to a} [f(x)]^{p(x)}$ (c) $\lim_{x \to a} [h(x)]^{p(x)}$

(d) $\lim_{x \to a} [p(x)]^{f(x)}$ (e) $\lim_{x \to a} [p(x)]^{q(x)}$ (f) $\lim_{x \to a} \sqrt[q(x)]{p(x)}$

5–64 Find the limit. Use l'Hospital's Rule where appropriat there is a more elementary method, consider using it. If l'H Rule doesn't apply, explain why.

5. $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$

6. $\displaystyle\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

7. $\displaystyle\lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1}$

8. $\displaystyle\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$

9. $\displaystyle\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$

10. $\displaystyle\lim_{x \to 0} \frac{\sin 4x}{\tan 5x}$

12. $\displaystyle\lim_{t\to 0}\frac{e^{3t}-1}{t}$

14. $\displaystyle\lim_{\theta\to\pi/2}\frac{1-\sin\theta}{\csc\theta}$

16. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2}$

18. $\displaystyle\lim_{x\to\infty}\frac{\ln\ln x}{x}$

20. $\displaystyle\lim_{x\to 1}\frac{\ln x}{\sin\pi x}$

22. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x-\frac{1}{2}x^2}{x^3}$

24. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

26. $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{x^3}$

28. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}$

30. $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

32. $\displaystyle\lim_{x\to 0}\frac{x}{\tan^{-1}(4x)}$

34. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}$

36. $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$

38. $\displaystyle\lim_{x\to a^+}\frac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$

40. $\displaystyle\lim_{x\to-\infty}x^2e^x$

42. $\displaystyle\lim_{x\to 0^+}\sin x\ln x$

44. $\displaystyle\lim_{x\to\pi/4}(1-\tan x)\sec x$

46. $\displaystyle\lim_{x\to\infty}x\tan(1/x)$

48. $\displaystyle\lim_{x\to 0}(\csc x-\cot x)$

50. $\displaystyle\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

51. $\displaystyle\lim_{x\to\infty}(x-\ln x)$

52. $\displaystyle\lim_{x\to\infty}(xe^{1/x}-x)$

53. $\displaystyle\lim_{x\to 0^+}x^{x^2}$

54. $\displaystyle\lim_{x\to 0^+}(\tan 2x)^x$

55. $\displaystyle\lim_{x\to 0}(1-2x)^{1/x}$

56. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

57. $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$

58. $\displaystyle\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}$

59. $\displaystyle\lim_{x\to\infty}x^{1/x}$

60. $\displaystyle\lim_{x\to\infty}(e^x+x)^{1/x}$

61. $\displaystyle\lim_{x\to 0^+}(4x+1)^{\cot x}$

62. $\displaystyle\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

63. $\displaystyle\lim_{x\to 0^+}(\cos x)^{1/x^2}$

64. $\displaystyle\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

65–66 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

65. $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

66. $\displaystyle\lim_{x\to 0}\frac{5^x-4^x}{3^x-2^x}$

67–68 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$ to see that these ratios have the same limit as $x\to 0$. Also calculate the exact value of the limit.

67. $f(x)=e^x-1,\quad g(x)=x^3+4x$

68. $f(x)=2x\sin x,\quad g(x)=\sec x-1$

69–74 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 4.5.

69. $y=xe^{-x}$

70. $y=\dfrac{\ln x}{x^2}$

71. $y=xe^{-x^2}$

72. $y=e^x/x$

73. $y=x-\ln(1+x)$

74. $y=(x^2-3)e^{-x}$

CAS **75–77**

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x\to 0^+$ or as $x\to\infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

75. $f(x)=x^{-x}$

76. $f(x)=(\sin x)^{\sin x}$

77. $f(x)=x^{1/x}$

78. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

79. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

80. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

81. What happens if you try to use l'Hospital's Rule to evaluate

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$$

Evaluate the limit using another method.

82. If a metal ball with mass m is projected in water and the force of resistance is proportional to the square of the velocity, then the distance the ball travels in time t is

$$s(t) = \frac{m}{c} \ln \cosh \sqrt{\frac{gc}{mt}}$$

where c is a positive constant. Find $\lim_{c \to 0^+} s(t)$.

83. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

84. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

85. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

86. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 10 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

87. In Section 5.3 we investigated the Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$, which arises in the study of the diffraction of light waves. Evaluate

$$\lim_{x \to 0} \frac{S(x)}{x^3}$$

88. Suppose that the temperature in a long thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leq a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\, du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

Use l'Hospital's Rule to find this limit.

89. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

90. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = 0$$

91. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

92. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

93. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

94. Prove that

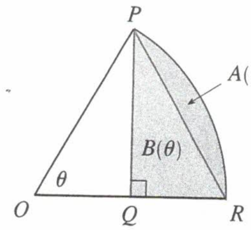
$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

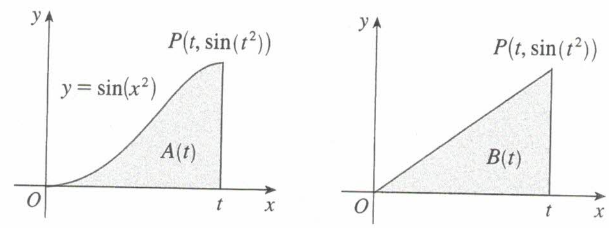
95. Prove that $\lim_{x \to 0^+} x^\alpha \ln x = 0$ for any $\alpha > 0$.

96. Evaluate $\lim_{x \to 0} \dfrac{1}{x^3} \displaystyle\int_0^x \sin(t^2)\, dt$.

97. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



98. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim_{t \to 0^+} A(t)/B(t)$.



99. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.
(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

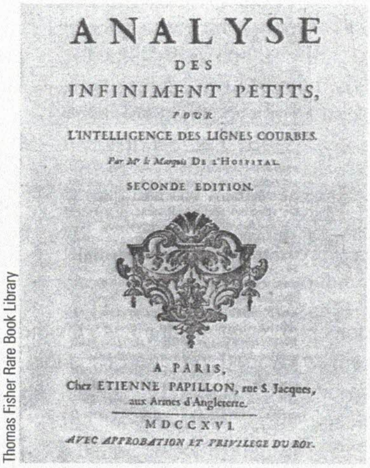
100. Let

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.
(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point $(0, 1)$ on the graph of f.
(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

WRITING PROJECT — THE ORIGINS OF L'HOSPITAL'S RULE

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits,* but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's



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mathematical discoveries. The details, including a translation of l'Hospital's letter to Bern proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by viding brief biographical details of both men (the dictionary edited by Gillispie [2] is a go source) and outline the business deal between them. Then give l'Hospital's statement of h which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notic l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms o ferentials. Compare their statement with the version of l'Hospital's Rule given in Section show that the two statements are essentially the same.

1. Howard Eves, *In Mathematical Circles (Volume 2: Quadrants III and IV)* (Boston: Pr Weber and Schmidt, 1969), pp. 20–22.
2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II an article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.
3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1 p. 484.
4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton versity Press, 1969), pp. 315–316.

7 REVIEW

CONCEPT CHECK

1. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
   (b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?
   (c) If f is a one-to-one function and $f'(f^{-1}(a)) \neq 0$, write a formula for $(f^{-1})'(a)$.
2. (a) What are the domain and range of the natural exponential function $f(x) = e^x$?
   (b) What are the domain and range of the natural logarithmic function $f(x) = \ln x$?
   (c) How are the graphs of these functions related? Sketch these graphs by hand, using the same axes.
   (d) If a is a positive number, $a \neq 1$, write an equation that expresses $\log_a x$ in terms of $\ln x$.
3. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
   (b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
   (c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range? Sketch its graph.
4. Write the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, and $\tanh x$.
5. State the derivative of each function.
   (a) $y = e^x$ (b) $y = a^x$ (c) $y = \ln x$
   (d) $y = \log_a x$ (e) $y = \sin^{-1}x$ (f) $y = \cos^{-1}$
   (g) $y = \tan^{-1}x$ (h) $y = \sinh x$ (i) $y = \cosh$
   (j) $y = \tanh x$ (k) $y = \sinh^{-1}x$ (l) $y = \cosh^{-}$
   (m) $y = \tanh^{-1}x$
6. (a) How is the number e defined?
   (b) Express e as a limit.
   (c) Why is the natural exponential function $y = e^x$ use often in calculus than the other exponential functio $y = a^x$?
   (d) Why is the natural logarithmic function $y = \ln x$ u often in calculus than the other logarithmic function $y = \log_a x$?
7. (a) Write a differential equation that expresses the law natural growth.
   (b) Under what circumstances is this an appropriate m population growth?
   (c) What are the solutions of this equation?
8. (a) What does l'Hospital's Rule say?
   (b) How can you use l'Hospital's Rule if you have a pr $f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$
   (c) How can you use l'Hospital's Rule if you have a di $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as x –
   (d) How can you use l'Hospital's Rule if you have a p $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$

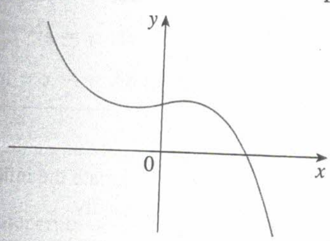
TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

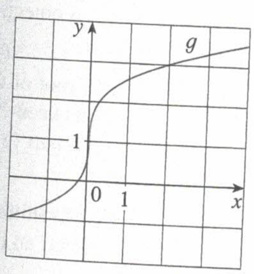
1. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$.
2. If f is one-to-one and differentiable, with domain $\mathbb{R}$, then $(f^{-1})'(6) = 1/f'(6)$.
3. The function $f(x) = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one.
4. $\tan^{-1}(-1) = 3\pi/4$
5. If $0 < a < b$, then $\ln a < \ln b$.
6. $\pi^{\sqrt{3}} = e^{\sqrt{3}\ln \pi}$
7. You can always divide by e^x.
8. If $a > 0$ and $b > 0$, then $\ln(a + b) = \ln a + \ln b$.
9. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.
10. $\dfrac{d}{dx}(10^x) = x10^{x-1}$
11. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$
12. The inverse function of $y = e^{3x}$ is $y = \frac{1}{3}\ln x$.
13. $\cos^{-1}x = \dfrac{1}{\cos x}$
14. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$
15. $\cosh x \geqslant 1$ for all x
16. $\ln \dfrac{1}{10} = -\displaystyle\int_1^{10} \frac{dx}{x}$
17. $\displaystyle\int_2^{16} \frac{dx}{x} = 3 \ln 2$
18. $\displaystyle\lim_{x \to \pi^-} \frac{\tan x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\sec^2 x}{\sin x} = \infty$

EXERCISES

1. The graph of f is shown. Is f one-to-one? Explain.



2. The graph of g is given.
   (a) Why is g one-to-one?
   (b) Estimate the value of $g^{-1}(2)$.
   (c) Estimate the domain of g^{-1}.
   (d) Sketch the graph of g^{-1}.



3. Suppose f is one-to-one, $f(7) = 3$, and $f'(7) = 8$. Find (a) $f^{-1}(3)$ and (b) $(f^{-1})'(3)$.

4. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

5–9 Sketch a rough graph of the function without using a calculator.

5. $y = 5^x - 1$
6. $y = -e^{-x}$
7. $y = -\ln x$
8. $y = \ln(x - 1)$
9. $y = 2 \arctan x$

10. Let $a > 1$. For large values of x, which of the functions $y = x^a$, $y = a^x$, and $y = \log_a x$ has the largest values and which has the smallest values?

11–12 Find the exact value of each expression.

11. (a) $e^{2\ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$
12. (a) $\ln e^{\pi}$ (b) $\tan\left(\arcsin \frac{1}{2}\right)$

13–20 Solve the equation for x.

13. $\ln x = \frac{1}{3}$
14. $e^x = \frac{1}{3}$
15. $e^{e^x} = 17$
16. $\ln(1 + e^{-x}) = 3$
17. $\ln(x + 1) + \ln(x - 1) = 1$
18. $\log_5(c^x) = d$
19. $\tan^{-1}x = 1$
20. $\sin x = 0.3$

21–47 Differentiate.

21. $f(t) = t^2 \ln t$
22. $g(t) = \dfrac{e^t}{1 + e^t}$
23. $h(\theta) = e^{\tan 2\theta}$
24. $h(u) = 10^{\sqrt{u}}$
25. $y = \ln|\sec 5x + \tan 5x|$
26. $y = e^{-t}(t^2 - 2t + 2)$
27. $y = e^{cx}(c \sin x - \cos x)$
28. $y = e^{mx} \cos nx$
29. $y = \ln(\sec^2 x)$
30. $y = \ln(x^2 e^x)$
31. $y = \dfrac{e^{1/x}}{x^2}$
32. $y = (\arcsin 2x)^2$

33. $y = 3^{x \ln x}$

34. $y = e^{\cos x} + \cos(e^x)$

35. $H(v) = v \tan^{-1} v$

36. $F(z) = \log_{10}(1 + z^2)$

37. $y = x \sinh(x^2)$

38. $y = (\cos x)^x$

39. $y = \ln \sin x - \frac{1}{2}\sin^2 x$

40. $y = \arctan\left(\arcsin \sqrt{x}\right)$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

42. $xe^y = y - 1$

43. $y = \ln(\cosh 3x)$

44. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x \tanh^{-1}\sqrt{x}$

47. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

---

48. Show that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x + 1)^2}{x^2 + 1}\right) = \frac{1}{(1 + x)(1 + x^2)}$$

49–52 Find f' in terms of g'.

49. $f(x) = e^{g(x)}$

50. $f(x) = g(e^x)$

51. $f(x) = \ln|g(x)|$

52. $f(x) = g(\ln x)$

53–54 Find $f^{(n)}(x)$.

53. $f(x) = 2^x$

54. $f(x) = \ln(2x)$

---

55. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

56. Find y' if $y = x + \arctan y$.

57–58 Find an equation of the tangent to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $y = x \ln x$, (e, e)

---

59. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

60. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

61. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

62. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.
(c) When is this rate equal to 0?

63–78 Evaluate the limit.

63. $\lim_{x\to\infty} e^{-3x}$

64. $\lim_{x\to 10^-} \ln(100 - x^2)$

65. $\lim_{x\to 3^-} e^{2/(x-3)}$

66. $\lim_{x\to\infty} \arctan(x^3 - x)$

67. $\lim_{x\to 0^+} \ln(\sinh x)$

68. $\lim_{x\to\infty} e^{-x}\sin x$

69. $\lim_{x\to\infty} \dfrac{1 + 2^x}{1 - 2^x}$

70. $\lim_{x\to\infty}\left(1 + \dfrac{4}{x}\right)^x$

71. $\lim_{x\to 0} \dfrac{\tan \pi x}{\ln(1 + x)}$

72. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2 + x}$

73. $\lim_{x\to 0} \dfrac{e^{4x} - 1 - 4x}{x^2}$

74. $\lim_{x\to\infty} \dfrac{e^{4x} - 1 - 4x}{x^2}$

75. $\lim_{x\to\infty} x^3 e^{-x}$

76. $\lim_{x\to 0^+} x^2 \ln x$

77. $\lim_{x\to 1^+}\left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x}\right)$

78. $\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x}$

79–84 Sketch the curve using the guidelines of Section 4.5.

79. $y = \tan^{-1}(1/x)$

80. $y = \sin^{-1}(1/x)$

81. $y = x \ln x$

82. $y = e^{2x - x^2}$

83. $y = xe^{-2x}$

84. $y = x + \ln(x^2 + 1)$

---

85. Graph $f(x) = e^{-1/x^2}$ in a viewing rectangle that shows all t main aspects of this function. Estimate the inflection point Then use calculus to find them exactly.

86. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What h pens to the maximum and minimum points and the inflecti points as c changes? Illustrate your conclusions by graphin several members of the family.

87. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the veloc and acceleration of the object.

88. (a) Show that there is exactly one root of the equation $\ln x = 3 - x$ and that it lies between 2 and e.
(b) Find the root of the equation in part (a) correct to four decimal places.

89. A bacteria culture contains 200 cells initially and grows a rate proportional to its size. After half an hour the populat has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

90. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample aft 20 years.
(b) How long would it take for the mass to decay to 1 m

91. The biologist G. F. Gause conducted an experiment in the 1930s with the protozoan *Paramecium* and used the population function

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

to model his data, where t was measured in days. Use this model to determine when the population was increasing most rapidly.

92–105 Evaluate the integral.

92. $\int_0^4 \frac{1}{16 + t^2}\,dt$

93. $\int_0^1 ye^{-2y^2}\,dy$

94. $\int_2^3 \frac{dr}{1 + 2r}$

95. $\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

96. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

98. $\int \frac{\cos(\ln x)}{x}\,dx$

99. $\int \frac{x + 1}{x^2 + 2x}\,dx$

100. $\int \frac{\csc^2 x}{1 + \cot x}\,dx$

101. $\int \tan x \ln(\cos x)\,dx$

102. $\int \frac{x}{\sqrt{1 - x^4}}\,dx$

103. $\int 2^{\tan\theta} \sec^2\theta\,d\theta$

104. $\int \sinh au\,du$

105. $\int \left(\frac{1 - x}{x}\right)^2 dx$

106–108 Use properties of integrals to prove the inequality.

106. $\int_0^1 \sqrt{1 + e^{2x}}\,dx \geqslant e - 1$

107. $\int_0^1 e^x \cos x\,dx \leqslant e - 1$

108. $\int_0^1 x \sin^{-1}x\,dx \leqslant \pi/4$

109–110 Find $f'(x)$.

109. $f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\,ds$

110. $f(x) = \int_{\ln x}^{2x} e^{-t^2}\,dt$

111. Find the average value of the function $f(x) = 1/x$ on the interval $[1, 4]$.

112. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.

113. Find the volume of the solid obtained by rotating about the y-axis the region under the curve $y = 1/(1 + x^4)$ from $x = 0$ to $x = 1$.

114. If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

115. If $f(x) = \ln x + \tan^{-1}x$, find $(f^{-1})'(\pi/4)$.

116. What is the area of the largest rectangle in the first quadrant with two sides on the axes and one vertex on the curve $y = e^{-x}$?

117. What is the area of the largest triangle in the first quadrant with two sides on the axes and the third side tangent to the curve $y = e^{-x}$?

118. Evaluate $\int_0^1 e^x\,dx$ without using the Fundamental Theorem of Calculus. [*Hint:* Use the definition of a definite integral with right endpoints, sum a geometric series, and then use l'Hospital's Rule.]

119. If $F(x) = \int_a^b t^x\,dt$, where $a, b > 0$, then, by the Fundamental Theorem,

$$F(x) = \frac{b^{x+1} - a^{x+1}}{x + 1} \qquad x \neq -1$$

$$F(-1) = \ln b - \ln a$$

Use l'Hospital's Rule to show that F is continuous at -1.

120. Show that

$$\cos\{\arctan[\sin(\operatorname{arccot} x)]\} = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$$

121. If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

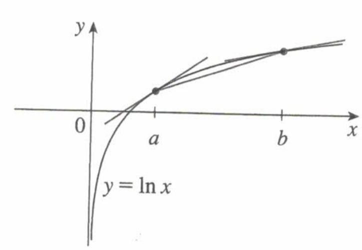
122. (a) Show that $\ln x < x - 1$ for $x > 0$, $x \neq 1$.
(b) Show that, for $x > 0$, $x \neq 1$,

$$\frac{x - 1}{x} < \ln x$$

(c) Deduce *Napier's Inequality:*

$$\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

if $b > a > 0$.
(d) Give a geometric proof of Napier's Inequality by comparing the slopes of the three lines shown in the figure.



(e) Give another proof of Napier's Inequality by applying Property 8 of integrals (see Section 5.2) to $\int_a^b (1/x)\,dx$.

PROBLEMS PLUS

Cover up the solution to the example and try it yourself.

EXAMPLE 1 For what values of c does the equation $\ln x = cx^2$ have exactly one solution?

SOLUTION One of the most important principles of problem solving is to draw a diagram, even if the problem as stated doesn't explicitly mention a geometric situation. Our present problem can be reformulated geometrically as follows: For what values of c does the curve $y = \ln x$ intersect the curve $y = cx^2$ in exactly one point?

Let's start by graphing $y = \ln x$ and $y = cx^2$ for various values of c. We know that, for $c \neq 0$, $y = cx^2$ is a parabola that opens upward if $c > 0$ and downward if $c < 0$. Figure 1 shows the parabolas $y = cx^2$ for several positive values of c. Most of them don't intersect $y = \ln x$ at all and one intersects twice. We have the feeling that there must be a value of c (somewhere between 0.1 and 0.3) for which the curves intersect exactly once, as in Figure 2.

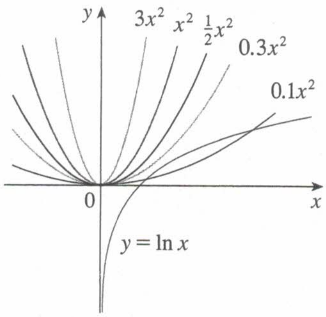


FIGURE 1

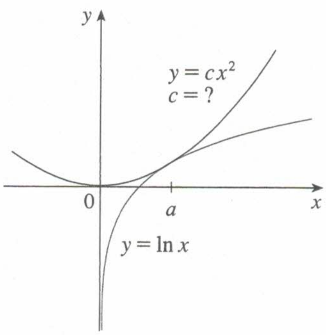


FIGURE 2

To find that particular value of c, we let a be the x-coordinate of the single point of intersection. In other words, $\ln a = ca^2$, so a is the unique solution of the given equation. We see from Figure 2 that the curves just touch, so they have a common tangent line when $x = a$. That means the curves $y = \ln x$ and $y = cx^2$ have the same slope when $x = a$. Therefore

$$\frac{1}{a} = 2ca$$

Solving the equations $\ln a = ca^2$ and $1/a = 2ca$, we get

$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2}$$

Thus $a = e^{1/2}$ and

$$c = \frac{\ln a}{a^2} = \frac{\ln e^{1/2}}{e} = \frac{1}{2e}$$

For negative values of c we have the situation illustrated in Figure 3: All parabolas $y = cx^2$ with negative values of c intersect $y = \ln x$ exactly once. And let's not forget about $c = 0$: The curve $y = 0x^2 = 0$ is just the x-axis, which intersects $y = \ln x$ exactly once.

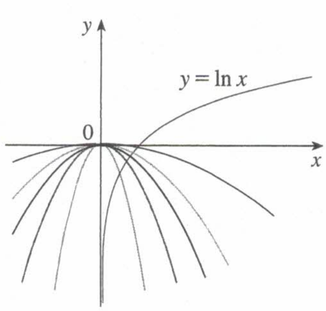


FIGURE 3

To summarize, the required values of c are $c = 1/(2e)$ and $c \leqslant 0$. □

PROBLEMS PLUS

PROBLEMS

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Prove that $\log_2 5$ is an irrational number.

3. Show that

$$\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

4. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

5. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

6. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\,dy$.

7. Show that $f(x) = \int_1^x \sqrt{1 + t^3}\,dt$ is one-to-one and find $(f^{-1})'(0)$.

8. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}}\arctan\frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \dfrac{1}{a + \cos x}$.

9. For what value of a is the following equation true?

$$\lim_{x \to \infty}\left(\frac{x + a}{x - a}\right)^x = e$$

10. Sketch the set of all points (x, y) such that $|x + y| \leqslant e^x$.

11. Prove that $\cosh(\sinh x) < \sinh(\cosh x)$ for all x.

12. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \geqslant e^2$$

13. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

14. For which positive numbers a is it true that $a^x \geqslant 1 + x$ for all x?

15. For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

16. For what values of c does the curve $y = cx^3 + e^x$ have inflection points?