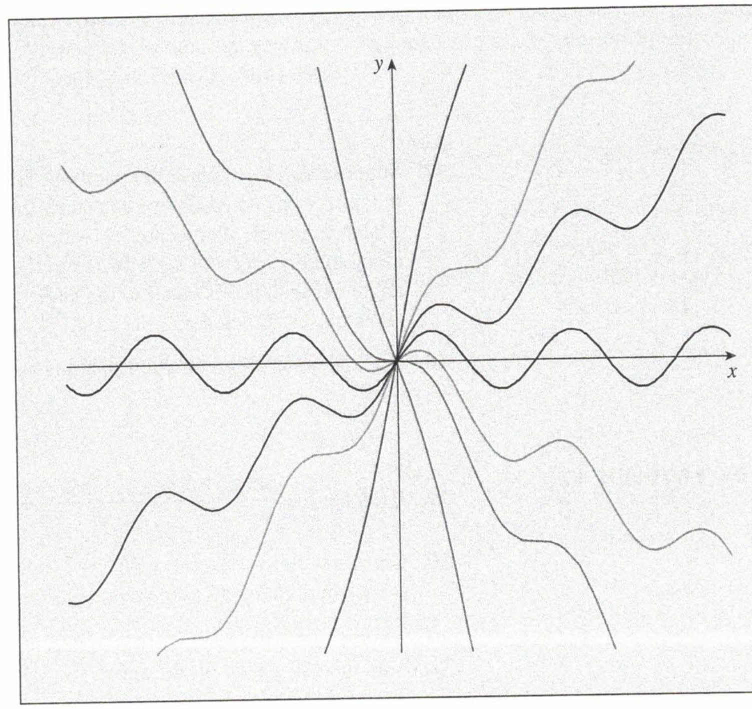
4

APPLICATIONS OF DIFFERENTIATION



Calculus reveals all the important aspects of graphs of functions. Members of the family of functions $f(x) = cx + \sin x$ are illustrated.

We have already investigated some of the applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. Here we learn how derivatives affect the shape of a graph of a function and, in particular, how they help us locate maximum and minimur values of functions. Many practical problems require us to minimize a cost or maximi an area or somehow find the best possible outcome of a situation. In particular, we wi be able to investigate the optimal shape of a can and to explain the location of rainbov in the sky.

4.1 MAXIMUM AND MINIMUM VALUES

Some of the most important applications of differential calculus are *optimization problems,* in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle? (This is an important question to the astronauts who have to withstand the effects of acceleration.)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maximum and minimum values.

1 DEFINITION A function f has an **absolute maximum** (or **global maximum**) at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the **maximum value** of f on D. Similarly, f has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the **minimum value** of f on D. The maximum and minimum values of f are called the **extreme values** of f.

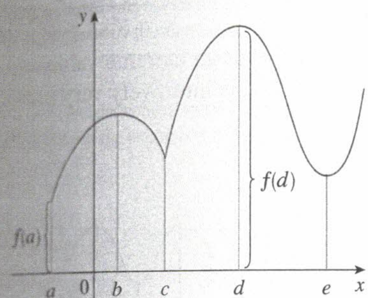


FIGURE 1
Minimum value $f(a)$, maximum value $f(d)$

Figure 1 shows the graph of a function f with absolute maximum at d and absolute minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point. If we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of f. Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \leqslant f(x)$ for x near c [in the interval (b, d), for instance]. The function f also has a local minimum at e. In general, we have the following definition.

2 DEFINITION A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geqslant f(x)$ when x is near c. [This means that $f(c) \geqslant f(x)$ for all x in some open interval containing c.] Similarly, f has a **local minimum** at c if $f(c) \leqslant f(x)$ when x is near c.

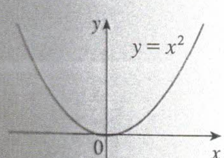


FIGURE 2
Minimum value 0, no maximum

EXAMPLE 1 The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times, since $\cos 2n\pi = 1$ for any integer n and $-1 \leqslant \cos x \leqslant 1$ for all x. Likewise, $\cos(2n + 1)\pi = -1$ is its minimum value, where n is any integer. □

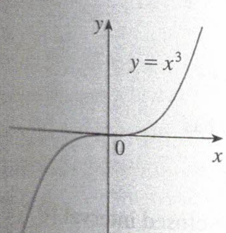


FIGURE 3
No minimum, no maximum

EXAMPLE 2 If $f(x) = x^2$, then $f(x) \geqslant f(0)$ because $x^2 \geqslant 0$ for all x. Therefore $f(0) = 0$ is the absolute (and local) minimum value of f. This corresponds to the fact that the origin is the lowest point on the parabola $y = x^2$. (See Figure 2.) However, there is no highest point on the parabola and so this function has no maximum value. □

EXAMPLE 3 From the graph of the function $f(x) = x^3$, shown in Figure 3, we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either. □

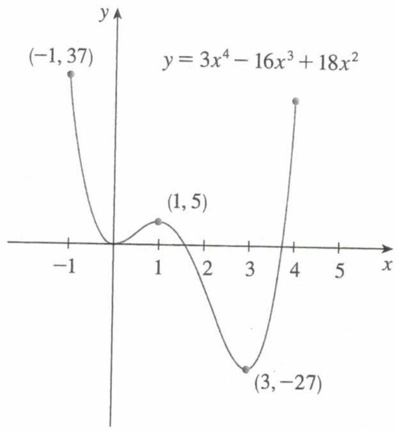


FIGURE 4

EXAMPLE 4 The graph of the function

$$f(x) = 3x^4 - 16x^3 + 18x^2 \qquad -1 \leqslant x \leqslant 4$$

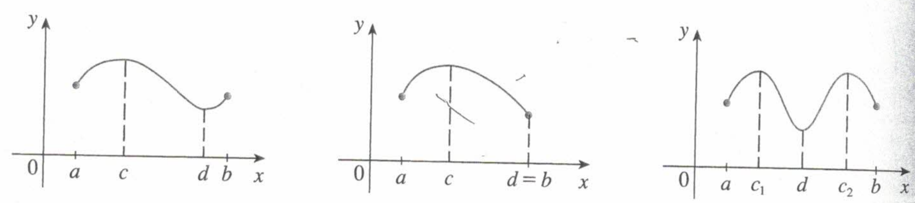
is shown in Figure 4. You can see that $f(1) = 5$ is a local maximum, whereas the absolute maximum is $f(-1) = 37$. (This absolute maximum is not a local maximum because it occurs at an endpoint.) Also, $f(0) = 0$ is a local minimum and $f(3) = -27$ is both a local and an absolute minimum. Note that f has neither a local nor an absolute maximum at $x = 4$. □

We have seen that some functions have extreme values, whereas others do not. The following theorem gives conditions under which a function is guaranteed to possess extreme values.

3 THE EXTREME VALUE THEOREM If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

The Extreme Value Theorem is illustrated in Figure 5. Note that an extreme value can be taken on more than once. Although the Extreme Value Theorem is intuitively very plausible, it is difficult to prove and so we omit the proof.

FIGURE 5



Figures 6 and 7 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

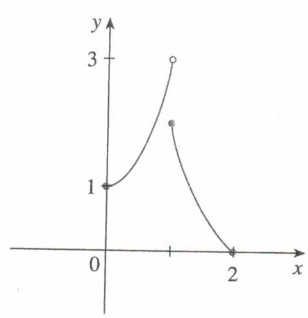


FIGURE 6
This function has minimum value $f(2) = 0$, but no maximum value.

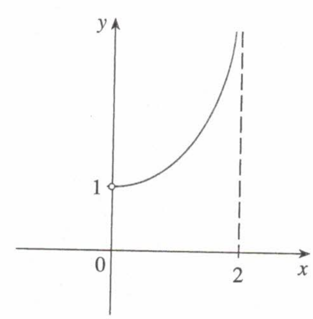


FIGURE 7
This continuous function g has no maximum or minimum.

The function f whose graph is shown in Figure 6 is defined on the closed interval $[0, 2]$ but has no maximum value. (Notice that the range of f is $[0, 3)$. The function takes on values arbitrarily close to 3, but never actually attains the value 3.) This does not contradict the Extreme Value Theorem because f is not continuous. [Nonetheless, a discontinuous function *could* have maximum and minimum values. See Exercise 13(b).]

The function g shown in Figure 7 is continuous on the open interval $(0, 2)$ but has neither a maximum nor a minimum value. [The range of g is $(1, \infty)$. The function takes on arbitrarily large values.] This does not contradict the Extreme Value Theorem because the interval $(0, 2)$ is not closed.

The Extreme Value Theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these extreme values. We start by looking for local extreme values.

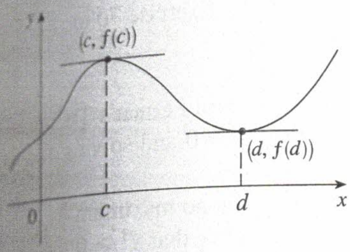


FIGURE 8

Figure 8 shows the graph of a function f with a local maximum at c and a local minimum at d. It appears that at the maximum and minimum points the tangent lines are horizontal and therefore each has slope 0. We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true for differentiable functions.

■ Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

> **4 FERMAT'S THEOREM** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

PROOF Suppose, for the sake of definiteness, that f has a local maximum at c. Then, according to Definition 2, $f(c) \geqslant f(x)$ if x is sufficiently close to c. This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geqslant f(c + h)$$

and therefore

$$f(c + h) - f(c) \leqslant 0 \tag{5}$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c + h) - f(c)}{h} \leqslant 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \leqslant \lim_{h \to 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h}$$

and so we have shown that $f'(c) \leqslant 0$.

If $h < 0$, then the direction of the inequality (5) is reversed when we divide by h:

$$\frac{f(c + h) - f(c)}{h} \geqslant 0 \qquad h < 0$$

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \geqslant 0$$

We have shown that $f'(c) \geqslant 0$ and also that $f'(c) \leqslant 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner, or we could use Exercise 70 to deduce it from the case we have just proved (see Exercise 71).

The following examples caution us against reading too much into Fermat's Theorem. We can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for

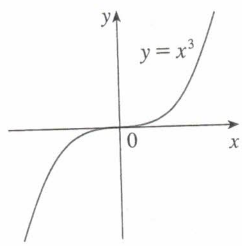


FIGURE 9
If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

EXAMPLE 5 If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in Figure 9. (Or observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x$ has a horizontal tangent at (0, 0). Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

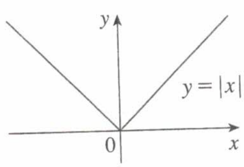


FIGURE 10
If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

EXAMPLE 6 The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, as was shown in Example 5 in Section 3.2, $f'(0)$ does not exist. (See Figure 10.)

WARNING Examples 5 and 6 show that we must be careful when using Ferma Theorem. Example 5 demonstrates that even when $f'(c) = 0$ there need not be a maximu or minimum at c. (In other words, the converse of Fermat's Theorem is false in genera Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as Example 6).

Fermat's Theorem does suggest that we should at least *start* looking for extreme valu of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers given a special name.

> **6 DEFINITION** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Figure 11 shows a graph of the function f in Example 7. It supports our answer because there is a horizontal tangent when $x = 1.5$ and a vertical tangent when $x = 0$.

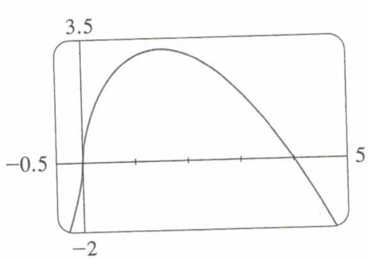


FIGURE 11

EXAMPLE 7 Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

SOLUTION The Product Rule gives

$$f'(x) = x^{3/5}(-1) + (4 - x)\left(\tfrac{3}{5}x^{-2/5}\right) = -x^{3/5} + \frac{3(4 - x)}{5x^{2/5}}$$

$$= \frac{-5x + 3(4 - x)}{5x^{2/5}} = \frac{12 - 8x}{5x^{2/5}}$$

[The same result could be obtained by first writing $f(x) = 4x^{3/5} - x^{8/5}$.] Therefore $f'(x) = 0$ if $12 - 8x = 0$, that is, $x = \frac{3}{2}$, and $f'(x)$ does not exist when $x = 0$. Thus t critical numbers are $\frac{3}{2}$ and 0.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows (com Definition 6 with Theorem 4):

> **7** If f has a local maximum or minimum at c, then c is a critical number of f.

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local [in which case it occurs at a critical number by (7)] or it occurs at an endpoint of the interval. Thus the following three-step procedure always works.

> **THE CLOSED INTERVAL METHOD** To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:
>
> **1.** Find the values of f at the critical numbers of f in (a, b).
>
> **2.** Find the values of f at the endpoints of the interval.
>
> **3.** The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

V EXAMPLE 8 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\tfrac{1}{2} \leqslant x \leqslant 4$$

SOLUTION Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $\left(-\frac{1}{2}, 4\right)$. The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

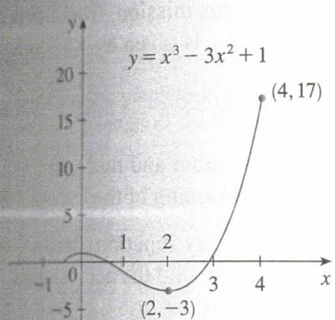


FIGURE 12

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. The graph of f is sketched in Figure 12. □

If you have a graphing calculator or a computer with graphing software, it is possible to estimate maximum and minimum values very easily. But, as the next example shows, calculus is needed to find the *exact* values.

EXAMPLE 9

(a) Use a graphing device to estimate the absolute minimum and maximum values of the function $f(x) = x - 2\sin x$, $0 \leqslant x \leqslant 2\pi$.

(b) Use calculus to find the exact minimum and maximum values.

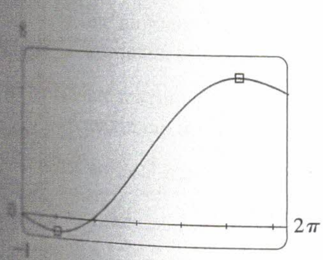


FIGURE 13

SOLUTION

(a) Figure 13 shows a graph of f in the viewing rectangle $[0, 2\pi]$ by $[-1, 8]$. By moving the cursor close to the maximum point, we see that the y-coordinates don't change very much in the vicinity of the maximum. The absolute maximum value is about 6.97 and it occurs when $x \approx 5.2$. Similarly, by moving the cursor close to the minimum point, we see that the absolute minimum value is about -0.68 and it occurs when $x \approx 1.0$. It is possible to get more accurate estimates by zooming in toward the maximum and minimum points, but instead let's use calculus.

(b) The function $f(x) = x - 2\sin x$ is continuous on $[0, 2\pi]$. Since $f'(x) = 1 - 2\cos x$, we have $f'(x) = 0$ when $\cos x = \frac{1}{2}$ and this occurs when $x = \pi/3$ or $5\pi/3$. The values of f at these critical points are

$$f(\pi/3) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} \approx -0.684853$$

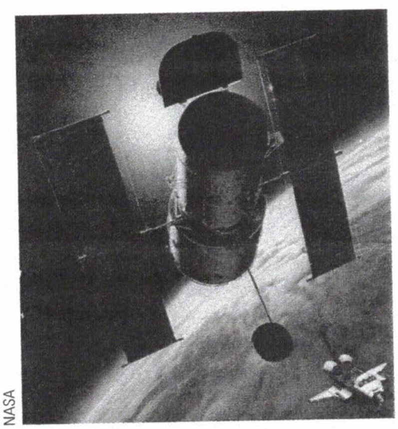
and

$$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.968039$$

The values of f at the endpoints are

$$f(0) = 0 \qquad \text{and} \qquad f(2\pi) = 2\pi \approx 6.28$$

Comparing these four numbers and using the Closed Interval Method, we see that the absolute minimum value is $f(\pi/3) = \pi/3 - \sqrt{3}$ and the absolute maximum value is $f(5\pi/3) = 5\pi/3 + \sqrt{3}$. The values from part (a) serve as a check on our work. □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EXAMPLE 10 The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery*. A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters.

SOLUTION We are asked for the extreme values not of the given velocity function, but rather of the acceleration function. So we first need to differentiate to find the acceleration:

$$a(t) = v'(t) = \frac{d}{dt}(0.001302t^3 - 0.09029t^2 + 23.61t - 3.083)$$

$$= 0.003906t^2 - 0.18058t + 23.61$$

We now apply the Closed Interval Method to the continuous function a on the interval $0 \leqslant t \leqslant 126$. Its derivative is

$$a'(t) = 0.007812t - 0.18058$$

The only critical number occurs when $a'(t) = 0$:

$$t_1 = \frac{0.18058}{0.007812} \approx 23.12$$

Evaluating $a(t)$ at the critical number and at the endpoints, we have

$$a(0) = 23.61 \qquad a(t_1) \approx 21.52 \qquad a(126) \approx 62.87$$

So the maximum acceleration is about 62.87 ft/s^2 and the minimum acceleration is about 21.52 ft/s^2.

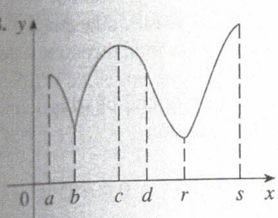
4.1 EXERCISES

1. Explain the difference between an absolute minimum and a local minimum.

2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
(a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
(b) What steps would you take to find those maximum and minimum values?

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.



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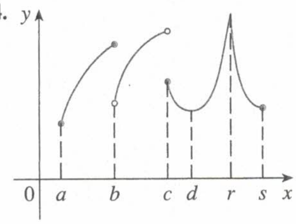

5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

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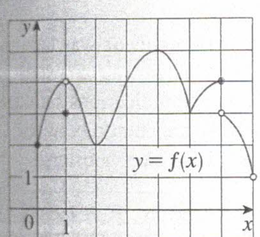

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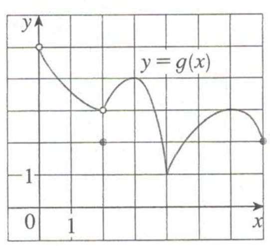


7–10 Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4

8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4

9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

10. f has no local maximum or minimum, but 2 and 4 are critical numbers

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 8 - 3x, \quad x \geqslant 1$

16. $f(x) = 3 - 2x, \quad x \leqslant 5$

17. $f(x) = x^2, \quad 0 < x < 2$

18. $f(x) = x^2, \quad 0 < x \leqslant 2$

19. $f(x) = x^2, \quad 0 \leqslant x < 2$

20. $f(x) = x^2, \quad 0 \leqslant x \leqslant 2$

21. $f(x) = x^2, \quad -3 \leqslant x \leqslant 2$

22. $f(x) = 1 + (x + 1)^2, \quad -2 \leqslant x < 5$

23. $f(t) = 1/t, \quad 0 < t < 1$

24. $f(t) = \cos t, \quad -3\pi/2 \leqslant t \leqslant 3\pi/2$

25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = 1 - x^3$

27. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leqslant x < 2 \\ 2x - 4 & \text{if } 2 \leqslant x \leqslant 3 \end{cases}$

28. $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \leqslant x < 0 \\ 2x - 1 & \text{if } 0 \leqslant x \leqslant 2 \end{cases}$

29–42 Find the critical numbers of the function.

29. $f(x) = 5x^2 + 4x$

30. $f(x) = x^3 + x^2 - x$

31. $f(x) = x^3 + 3x^2 - 24x$

32. $f(x) = x^3 + x^2 + x$

33. $s(t) = 3t^4 + 4t^3 - 6t^2$

34. $g(t) = |3t - 4|$

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

36. $h(p) = \dfrac{p - 1}{p^2 + 4}$

37. $h(t) = t^{3/4} - 2t^{1/4}$

38. $g(x) = \sqrt{1 - x^2}$

39. $F(x) = x^{4/5}(x - 4)^2$

40. $g(x) = x^{1/3} - x^{-2/3}$

41. $f(\theta) = 2\cos\theta + \sin^2\theta$

42. $g(\theta) = 4\theta - \tan\theta$

43–44 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

43. $f'(x) = 1 + \dfrac{210 \sin x}{x^2 - 6x + 10}$

44. $f'(x) = \dfrac{100 \cos^2 x}{10 + x^2} - 1$

45–56 Find the absolute maximum and absolute minimum values of f on the given interval.

45. $f(x) = 3x^2 - 12x + 5, \quad [0, 3]$

46. $f(x) = x^3 - 3x + 1, \quad [0, 3]$

47. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

48. $f(x) = x^3 - 6x^2 + 9x + 2, \quad [-1, 4]$

49. $f(x) = x^4 - 2x^2 + 3, \quad [-2, 3]$

50. $f(x) = (x^2 - 1)^3, \quad [-1, 2]$

51. $f(x) = \dfrac{x}{x^2 + 1}, \quad [0, 2]$

52. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}, \quad [-4, 4]$

53. $f(t) = t\sqrt{4 - t^2}, \quad [-1, 2]$

54. $f(t) = \sqrt[3]{t}\,(8 - t), \quad [0, 8]$

55. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

56. $f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4]$

57. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leqslant x \leqslant 1$.

58. Use a graph to estimate the critical numbers of $f(x) = |x^3 - 3x^2 + 2|$ correct to one decimal place.

59–62

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

59. $f(x) = x^5 - x^3 + 2, \quad -1 \leqslant x \leqslant 1$

60. $f(x) = x^4 - 3x^3 + 3x^2 - x, \quad 0 \leqslant x \leqslant 2$

61. $f(x) = x\sqrt{x - x^2}$

62. $f(x) = x - 2\cos x, \quad -2 \leqslant x \leqslant 0$

63. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

64. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

65. A model for the US average price of a pound of white sugar from 1993 to 2003 is given by the function

$$S(t) = -0.00003237t^5 + 0.0009037t^4 - 0.008956t^3 + 0.03629t^2 - 0.04458t + 0.4074$$

where t is measured in years since August of 1993. Estimate the times when sugar was cheapest and most expensive during the period 1993–2003.

66. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

67. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \tfrac{1}{2}r_0 \leqslant r \leqslant r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0, r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

68. Show that 5 is a critical number of the function

$$g(x) = 2 + (x - 5)^3$$

but g does not have a local extreme value at 5.

69. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

70. If f has a local minimum value at c, show that the function $g(x) = -f(x)$ has a local maximum value at c.

71. Prove Fermat's Theorem for the case in which f has a local minimum at c.

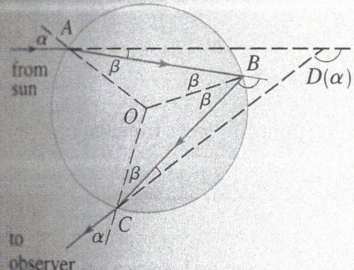
72. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.

(b) How many local extreme values can a cubic function have?

APPLIED PROJECT — THE CALCULUS OF RAINBOWS

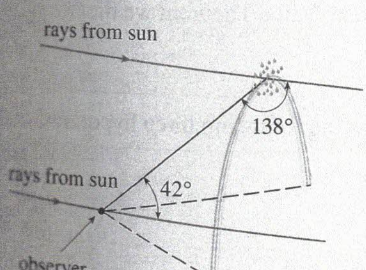
Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.



Formation of the primary rainbow

1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that $\sin\alpha = k\sin\beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The *angle of deviation* $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

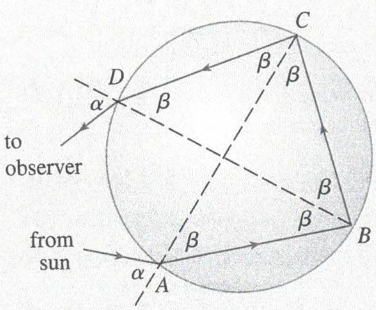
$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$



Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha \approx 59.4°$.

The significance of the minimum deviation is that when $\alpha \approx 59.4°$ we have $D'(\alpha) \approx 0$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure at the left shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. (This angle is called the *rainbow angle*.)

2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion.*) For red light the refractive index is $k \approx 1.3318$ whereas for violet light it is $k \approx 1.3435$. By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.



Formation of the secondary rainbow

3. Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D (see the figure). This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

and $D(\alpha)$ has a minimum value when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{8}}$$

Taking $k = \frac{4}{3}$, show that the minimum deviation is about 129° and so the rainbow angle for the secondary rainbow is about 51°, as shown in the figure.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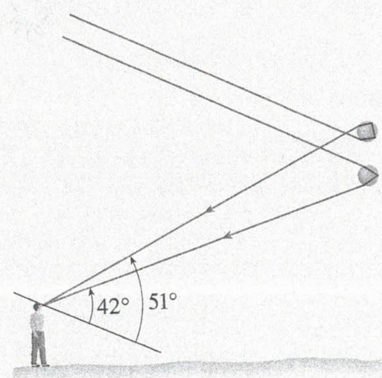


4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.

4.2 THE MEAN VALUE THEOREM

We will see that many of the results of this chapter depend on one central fact, which i called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need th following result.

Rolle's Theorem was first published in 1691 by the French mathematician Michel Rolle (1652–1719) in a book entitled *Méthode pour résoudre les égalitéz.* He was a vocal critic of the methods of his day and attacked calculus as being a "collection of ingenious fallacies." Later, however, he became convinced of the essential correctness of the methods of calculus.

ROLLE'S THEOREM Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus Rolle's Theorem is plausible.

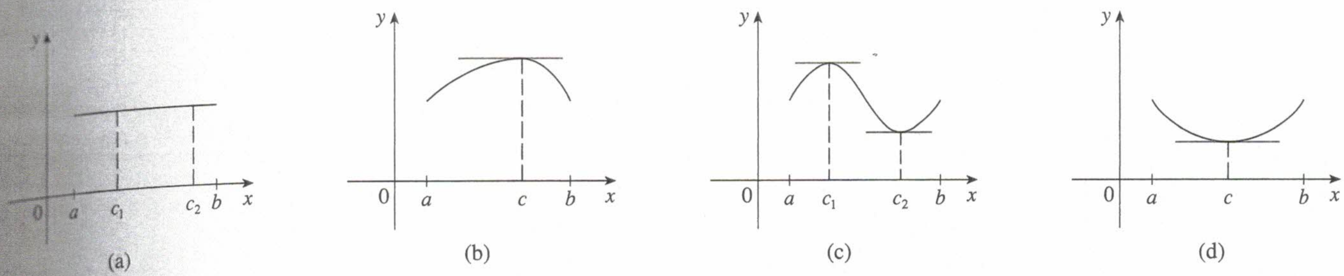


FIGURE 1

■ Take cases

PROOF There are three cases:

CASE I ■ $f(x) = k$, a constant
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

CASE II ■ $f(x) > f(a)$ for some x in (a, b) [as in Figure 1(b) or (c)]
By the Extreme Value Theorem (which we can apply by hypothesis 1), f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore $f'(c) = 0$ by Fermat's Theorem.

CASE III ■ $f(x) < f(a)$ for some x in (a, b) [as in Figure 1(c) or (d)]
By the Extreme Value Theorem, f has a minimum value in $[a, b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a, b). Again $f'(c) = 0$ by Fermat's Theorem. □

EXAMPLE 1 Let's apply Rolle's Theorem to the position function $s = f(t)$ of a moving object. If the object is in the same place at two different instants $t = a$ and $t = b$, then $f(a) = f(b)$. Rolle's Theorem says that there is some instant of time $t = c$ between a and b when $f'(c) = 0$; that is, the velocity is 0. (In particular, you can see that this is true when a ball is thrown directly upward.) □

■ Figure 2 shows a graph of the function $f(x) = x^3 + x - 1$ discussed in Example 2. Rolle's Theorem shows that, no matter how much we enlarge the viewing rectangle, we can never find a second x-intercept.

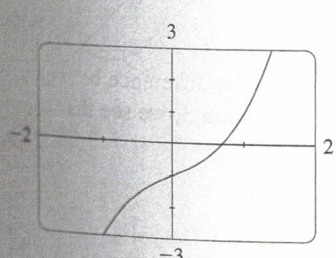


FIGURE 2

EXAMPLE 2 Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

SOLUTION First we use the Intermediate Value Theorem (2.5.10) to show that a root exists. Let $f(x) = x^3 + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem states that there is a number c between 0 and 1 such that $f(c) = 0$. Thus the given equation has a root.

To show that the equation has no other real root, we use Rolle's Theorem and argue by contradiction. Suppose that it had two roots a and b. Then $f(a) = 0 = f(b)$ and, since f is a polynomial, it is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$. But

$$f'(x) = 3x^2 + 1 \geqslant 1 \qquad \text{for all } x$$

(since $x^2 \geqslant 0$) so $f'(x)$ can never be 0. This gives a contradiction. Therefore the equation can't have two real roots. □

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

THE MEAN VALUE THEOREM Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$

■ The Mean Value Theorem is an example of what is called an existence theorem. Like the Intermediate Value Theorem, the Extreme Value Theorem, and Rolle's Theorem, it guarantees that there *exists* a number with a certain property, but it doesn't tell us how to find the number.

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions. The slope of the secant line AB is

3 $$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1. Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB. In other words, there is a point P where the tangent line is parallel to the secant line AB.

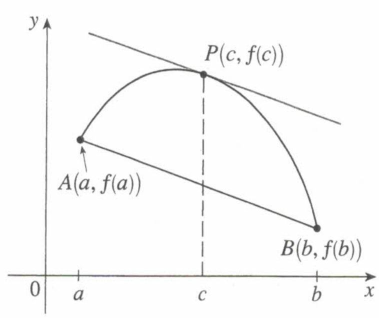


FIGURE 3

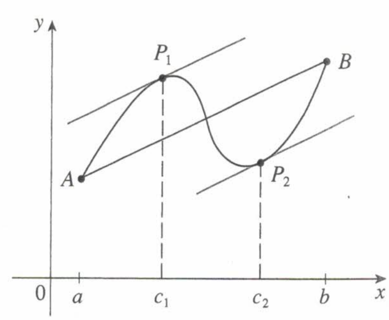


FIGURE 4

PROOF We apply Rolle's Theorem to a new function h defined as the difference between f and the function whose graph is the secant line AB. Using Equation 3, we see that the equation of the line AB can be written as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or as

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

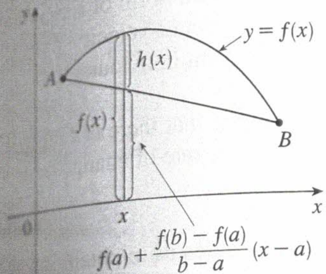


FIGURE 5

LAGRANGE AND THE MEAN VALUE THEOREM

The Mean Value Theorem was first formulated by Joseph-Louis Lagrange (1736–1813), born in Italy of a French father and an Italian mother. He was a child prodigy and became a professor in Turin at the tender age of 19. Lagrange made great contributions to number theory, theory of functions, theory of equations, and analytical and celestial mechanics. In particular, he applied calculus to the analysis of the stability of the solar system. At the invitation of Frederick the Great, he succeeded Euler at the Berlin Academy and, when Frederick died, Lagrange accepted King Louis XVI's invitation to Paris, where he was given apartments in the Louvre and became a professor at the Ecole Polytechnique. Despite all the trappings of luxury and fame, he was a kind and quiet man, living only for science.

So, as shown in Figure 5,

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{4}$$

First we must verify that h satisfies the three hypotheses of Rolle's Theorem.

1. The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.

2. The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable. In fact, we can compute h' directly from Equation 4:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

(Note that $f(a)$ and $[f(b) - f(a)]/(b - a)$ are constants.)

3.
$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$\begin{aligned} h(b) &= f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= f(b) - f(a) - [f(b) - f(a)] = 0 \end{aligned}$$

Therefore $h(a) = h(b)$.

Since h satisfies the hypotheses of Rolle's Theorem, that theorem says there is a number c in (a, b) such that $h'(c) = 0$. Therefore

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ □

V EXAMPLE 3 To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now $f(2) = 6$, $f(0) = 0$, and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2$$

which gives $c^2 = \frac{4}{3}$, that is, $c = \pm 2/\sqrt{3}$. But c must lie in $(0, 2)$, so $c = 2/\sqrt{3}$. Figure 6 illustrates this calculation: The tangent line at this value of c is parallel to the secant line OB. □

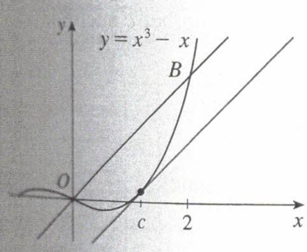


FIGURE 6

V EXAMPLE 4 If an object moves in a straight line with position function $s = f(t)$, then the average velocity between $t = a$ and $t = b$ is

$$\frac{f(b) - f(a)}{b - a}$$

and the velocity at $t = c$ is $f'(c)$. Thus the Mean Value Theorem (in the form of Equation 1) tells us that at some time $t = c$ between a and b the instantaneous velocity $f'(c)$ is equal to that average velocity. For instance, if a car traveled 180 km in 2 hours, then the speedometer must have read 90 km/h at least once.

In general, the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval. □

The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative. The next example provides an instance of this principle.

V EXAMPLE 5 Suppose that $f(0) = -3$ and $f'(x) \leqslant 5$ for all values of x. How large can $f(2)$ possibly be?

SOLUTION We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

so

$$f(2) = f(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leqslant 5$ for all x, so in particular we know that $f'(c) \leqslant 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leqslant 10$, so

$$f(2) = -3 + 2f'(c) \leqslant -3 + 10 = 7$$

The largest possible value for $f(2)$ is 7. □

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus. One of these basic facts is the following theorem. Others will be found in the following sections.

5 THEOREM If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

PROOF Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{6}$$

Since $f'(x) = 0$ for all x, we have $f'(c) = 0$, and so Equation 6 becomes

$$f(x_2) - f(x_1) = 0 \qquad \text{or} \qquad f(x_2) = f(x_1)$$

Therefore f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b). □

7 COROLLARY If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by Theorem 5, F is constant; that is, $f - g$ is constant. □

NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

4.2 EXERCISES

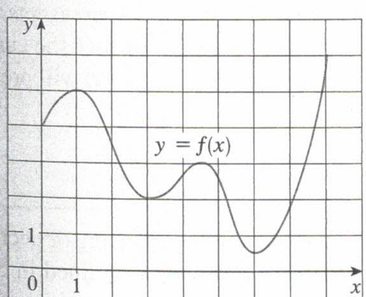
1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = 5 - 12x + 3x^2$, $[1, 3]$
2. $f(x) = x^3 - x^2 - 6x + 2$, $[0, 3]$
3. $f(x) = \sqrt{x} - \frac{1}{3}x$, $[0, 9]$
4. $f(x) = \cos 2x$, $[\pi/8, 7\pi/8]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

6. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



8. Use the graph of f given in Exercise 7 to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[1, 7]$.

9. (a) Graph the function $f(x) = x + 4/x$ in the viewing rectangle $[0, 10]$ by $[0, 10]$.
   (b) Graph the secant line that passes through the points $(1, 5)$ and $(8, 8.5)$ on the same screen with f.
   (c) Find the number c that satisfies the conclusion of the Mean Value Theorem for this function f and the interval $[1, 8]$. Then graph the tangent line at the point $(c, f(c))$ and notice that it is parallel to the secant line.

10. (a) In the viewing rectangle $[-3, 3]$ by $[-5, 5]$, graph the function $f(x) = x^3 - 2x$ and its secant line through the points $(-2, -4)$ and $(2, 4)$. Use the graph to estimate the x-coordinates of the points where the tangent line is parallel to the secant line.
    (b) Find the exact values of the numbers c that satisfy the conclusion of the Mean Value Theorem for the interval $[-2, 2]$ and compare with your answers to part (a).

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$
12. $f(x) = x^3 + x - 1$, $[0, 2]$
13. $f(x) = \sqrt[3]{x}$, $[0, 1]$
14. $f(x) = \dfrac{x}{x + 2}$, $[1, 4]$

15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

16. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

17. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

18. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

19. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

20. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

21. (a) Show that a polynomial of degree 3 has at most three real roots.
(b) Show that a polynomial of degree n has at most n real roots.

22. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.
(c) Can you generalize parts (a) and (b)?

23. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

24. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

25. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint*: Apply the Mea Value Theorem to the function $h = f - g$.]

27. Show that $\sqrt{1 + x} < 1 + \frac{1}{2}x$ if $x > 0$.

28. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

29. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

30. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to sho that $f(x) = cx + d$ for some constant d.

31. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

32. At 2:00 PM a car's speedometer reads 30 mi/h. At 2:10 PM reads 50 mi/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 120 mi/h^2.

33. Two runners start a race at the same time and finish in a ti Prove that at some time during the race they have the sam speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h the position functions of the two runners.]

34. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, tl f has at most one fixed point.

4.3 HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH

Many of the applications of calculus depend on our ability to deduce facts about a tion f from information concerning its derivatives. Because $f'(x)$ represents the slc the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve pro at each point. So it is reasonable to expect that information about $f'(x)$ will provide u: information about $f(x)$.

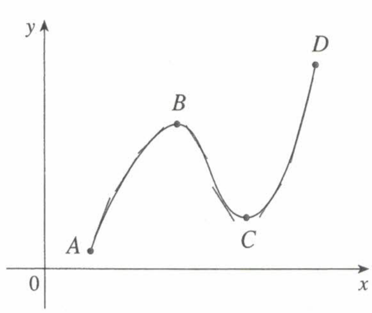


FIGURE 1

WHAT DOES f' SAY ABOUT f?

To see how the derivative of f can tell us where a function is increasing or decreasing at Figure 1. (Increasing functions and decreasing functions were defined in Sectior Between A and B and between C and D, the tangent lines have positive slope a $f'(x) > 0$. Between B and C, the tangent lines have negative slope and so $f'(x) < 0$. it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negati prove that this is always the case, we use the Mean Value Theorem.

■ Let's abbreviate the name of this test to the I/D Test.

INCREASING/DECREASING TEST

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF

(a) Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. According to the definition of an increasing function (page 20) we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on $[x_1, x_2]$. So, by the Mean Value Theorem there is a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{1}$$

Now $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$. Thus the right side of Equation 1 is positive, and so

$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_1) < f(x_2)$$

This shows that f is increasing.

Part (b) is proved similarly. □

V EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

SOLUTION $$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

To use the I/D Test we have to know where $f'(x) > 0$ and where $f'(x) < 0$. This depends on the signs of the three factors of $f'(x)$, namely, $12x$, $x - 2$, and $x + 1$. We divide the real line into intervals whose endpoints are the critical numbers -1, 0, and 2 and arrange our work in a chart. A plus sign indicates that the given expression is positive, and a minus sign indicates that it is negative. The last column of the chart gives the conclusion based on the I/D Test. For instance, $f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	−	−	−	−	decreasing on $(-\infty, -1)$
$-1 < x < 0$	−	−	+	+	increasing on $(-1, 0)$
$0 < x < 2$	+	−	+	−	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	increasing on $(2, \infty)$

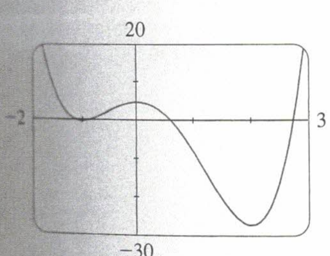


FIGURE 2

The graph of f shown in Figure 2 confirms the information in the chart. □

Recall from Section 4.1 that if f has a local maximum or minimum at c, then c must be a critical number of f (by Fermat's Theorem), but not every critical number gives rise to a maximum or a minimum. We therefore need a test that will tell us whether or not f has a local maximum or minimum at a critical number.

You can see from Figure 2 that $f(0) = 5$ is a local maximum value of f because f increases on $(-1, 0)$ and decreases on $(0, 2)$. Or, in terms of derivatives, $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 2$. In other words, the sign of $f'(x)$ changes from positive to negative at 0. This observation is the basis of the following test.

THE FIRST DERIVATIVE TEST Suppose that c is a critical number of a function f.

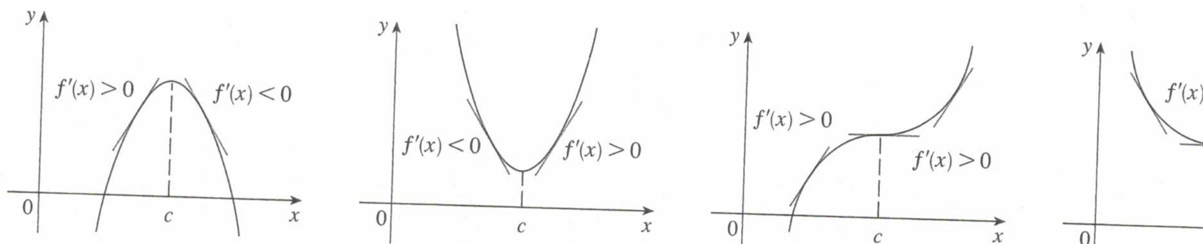
(a) If f' changes from positive to negative at c, then f has a local m

(b) If f' changes from negative to positive at c, then f has a local mi

(c) If f' does not change sign at c (for example, if f' is positive on b or negative on both sides), then f has no local maximum or mini

The First Derivative Test is a consequence of the I/D Test. In part (a), f the sign of $f'(x)$ changes from positive to negative at c, f is increasing to decreasing to the right of c. It follows that f has a local maximum at c.

It is easy to remember the First Derivative Test by visualizing diagra in Figure 3.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maxim

FIGURE 3

EXAMPLE 2 Find the local minimum and maximum values of the funct Example 1.

SOLUTION From the chart in the solution to Example 1 we see that $f'(x)$ cha ative to positive at -1, so $f(-1) = 0$ is a local minimum value by the Firs Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ i minimum value. As previously noted, $f(0) = 5$ is a local maximum value changes from positive to negative at 0.

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

SOLUTION To find the critical numbers of g, we differentiate:

$$g'(x) = 1 + 2\cos x$$

So $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ an Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and so we analyze g in the following table.

■ The + signs in the table come from the fact that $g'(x) > 0$ when $\cos x > -\frac{1}{2}$. From the graph of $y = \cos x$, this is true in the indicated intervals.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	+	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	−	decreasing on $(2\pi/3, 4$
$4\pi/3 < x < 2\pi$	+	increasing on $(4\pi/3, 2\pi$

Because $g'(x)$ changes from positive to negative at $2\pi/3$, the First Derivative Test tells us that there is a local maximum at $2\pi/3$ and the local maximum value is

$$g(2\pi/3) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \approx 3.83$$

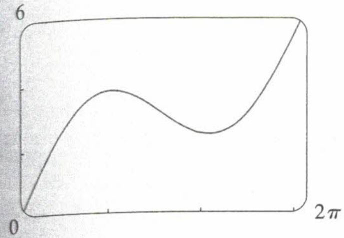


FIGURE 4
$y = x + 2\sin x$

Likewise, $g'(x)$ changes from negative to positive at $4\pi/3$ and so

$$g(4\pi/3) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \approx 2.46$$

is a local minimum value. The graph of g in Figure 4 supports our conclusion. □

WHAT DOES f'' SAY ABOUT f?

Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior? In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).

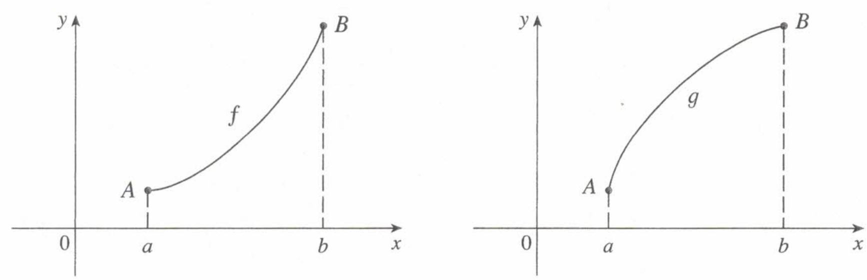


FIGURE 5 (a) (b)

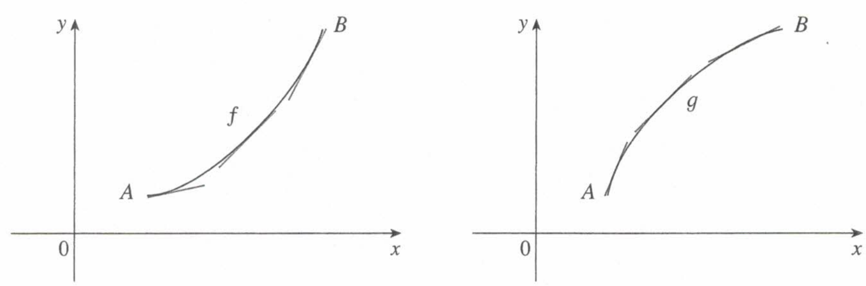
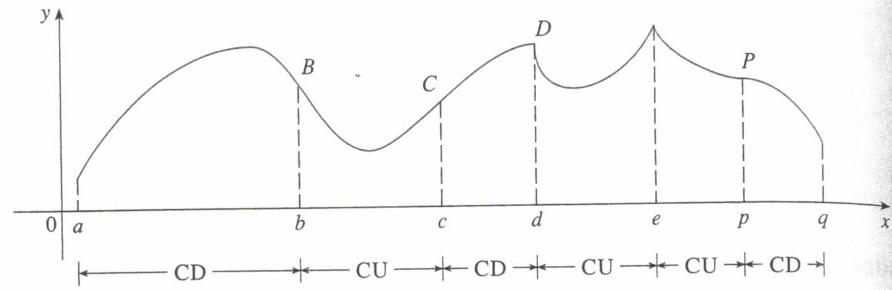


FIGURE 6 (a) Concave upward (b) Concave downward

> **DEFINITION** If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on t
intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), $(c,$
and (p, q).

FIGURE 7



Let's see how the second derivative helps determine the intervals of concavity. Looki
at Figure 6(a), you can see that, going from left to right, the slope of the tangent increas
This means that the derivative f' is an increasing function and therefore its derivative
is positive. Likewise, in Figure 6(b) the slope of the tangent decreases from left to rig
so f' decreases and therefore f'' is negative. This reasoning can be reversed and sugge
that the following theorem is true. A proof is given in Appendix F with the help of
Mean Value Theorem.

CONCAVITY TEST

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

EXAMPLE 4 Figure 8 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

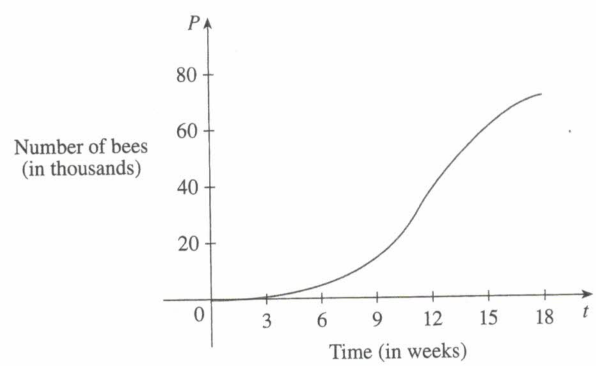


FIGURE 8

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of
increase of the population is initially very small, then gets larger until it reaches a ma
mum at about $t = 12$ weeks, and decreases as the population begins to level off. As t
population approaches its maximum value of about 75,000 (called the *carrying capa
ity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upwar
$(0, 12)$ and concave downward on $(12, 18)$.

In Example 4, the population curve changed from concave upward to concave downward at approximately the point (12, 38,000). This point is called an *inflection point* of the curve. The significance of this point is that the rate of population increase has its maximum value there. In general, an inflection point is a point where a curve changes its direction of concavity.

> **DEFINITION** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

For instance, in Figure 7, B, C, D, and P are the points of inflection. Notice that if a curve has a tangent at a point of inflection, then the curve crosses its tangent there.

In view of the Concavity Test, there is a point of inflection at any point where the second derivative changes sign.

EXAMPLE 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f(0) = 0, \quad f(2) = 3, \quad f(4) = 6, \quad f'(0) = f'(4) = 0$

(ii) $f'(x) > 0$ for $0 < x < 4, \quad f'(x) < 0$ for $x < 0$ and for $x > 4$

(iii) $f''(x) > 0$ for $x < 2, \quad f''(x) < 0$ for $x > 2$

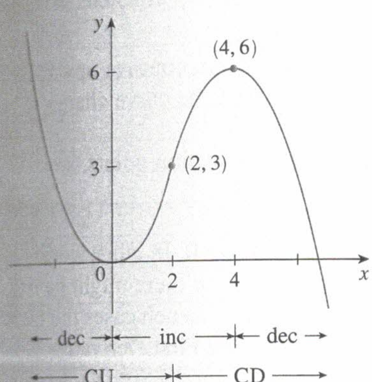


FIGURE 9

SOLUTION Condition (i) tells us that the graph has horizontal tangents at the points (0, 0) and (4, 6). Condition (ii) says that f is increasing on the interval (0, 4) and decreasing on the intervals $(-\infty, 0)$ and $(4, \infty)$. It follows from the I/D Test that $f(0) = 0$ is a local minimum and $f(4) = 6$ is a local maximum.

Condition (iii) says that the graph is concave upward on the interval $(-\infty, 2)$ and concave downward on $(2, \infty)$. Because the curve changes from concave upward to concave downward when $x = 2$, the point (2, 3) is an inflection point.

We use this information to sketch the graph of f in Figure 9. Notice that we made the curve bend upward when $x < 2$ and bend downward when $x > 2$. □

Another application of the second derivative is the following test for maximum and minimum values. It is a consequence of the Concavity Test.

> **THE SECOND DERIVATIVE TEST** Suppose f'' is continuous near c.
> (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
> (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

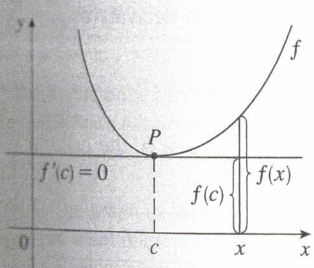


FIGURE 10
$f''(c) > 0$, f is concave upward

For instance, part (a) is true because $f''(x) > 0$ near c and so f is concave upward near c. This means that the graph of f lies *above* its horizontal tangent at c and so f has a local minimum at c. (See Figure 10.)

EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

SOLUTION If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0. [In fact, the expression for $f'(x)$ shows that f decreases to the left of 3 and increases to the right of 3.]

Since $f''(x) = 0$ when $x = 0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

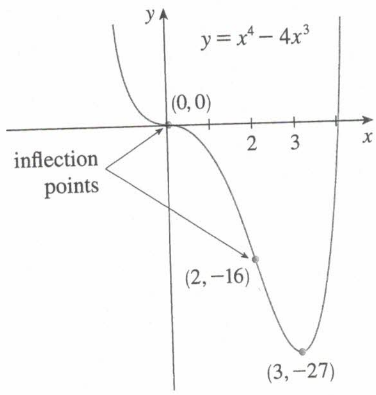


FIGURE 11

The point (0, 0) is an inflection point since the curve changes from concave upward to concave downward there. Also, $(2, -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

Using the local minimum, the intervals of concavity, and the inflection points, we sketch the curve in Figure 11. □

NOTE The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither (as in Example 6). This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

■ Try reproducing the graph in Figure 12 with a graphing calculator or computer. Some machines produce the complete graph, some produce only the portion to the right of the y-axis, and some produce only the portion between $x = 0$ and $x = 6$. For an explanation and cure, see Example 7 in Section 1.4. An equivalent expression that gives the correct graph is

$$y = (x^2)^{1/3} \cdot \frac{6 - x}{|6 - x|} |6 - x|^{1/3}$$

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$.

SOLUTION You can use the differentiation rules to check that the first two derivatives are

$$f'(x) = \frac{4 - x}{x^{1/3}(6 - x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6 - x)^{5/3}}$$

Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

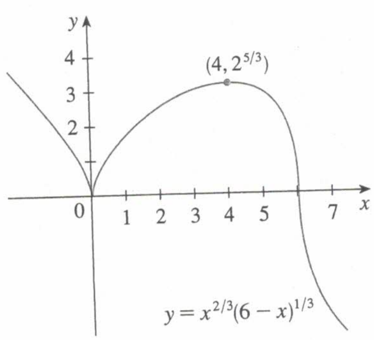


FIGURE 12

Interval	$4 - x$	$x^{1/3}$	$(6 - x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

To find the local extreme values we use the First Derivative Test. Since f' changes from negative to positive at 0, $f(0) = 0$ is a local minimum. Since f' changes from positive to negative at 4, $f(4) = 2^{5/3}$ is a local maximum. The sign of f' does not cha

TEC In Module 4.3 you can practice using graphical information about f' to determine the shape of the graph of f.

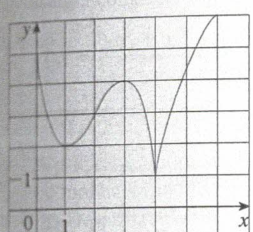
at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4, but not at 0 or 6 since f'' does not exist at either of these numbers.)

Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geq 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$. □

4.3 EXERCISES

1–2 Use the given graph of f to find the following.
(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



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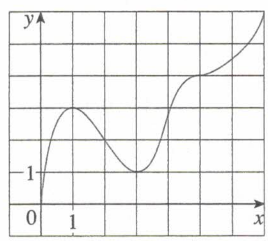
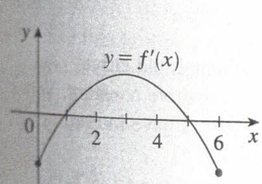


3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

5–6 The graph of the *derivative* f' of a function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?

5.



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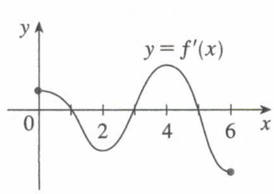
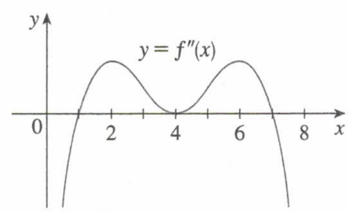
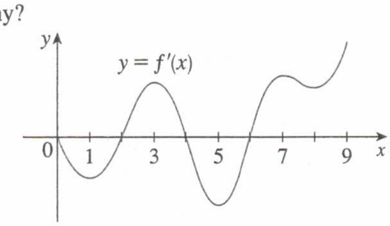


7. The graph of the second derivative f'' of a function f is shown. State the x-coordinates of the inflection points of f. Give reasons for your answers.



8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9–14
(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = 2x^3 + 3x^2 - 36x$

10. $f(x) = 4x^3 + 3x^2 - 6x + 1$

11. $f(x) = x^4 - 2x^2 + 3$

12. $f(x) = \dfrac{x^2}{x^2 + 3}$

13. $f(x) = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

14. $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leq x \leq 2\pi$

15–17 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

15. $f(x) = x^5 - 5x + 3$ **16.** $f(x) = \dfrac{x}{x^2 + 4}$

17. $f(x) = x + \sqrt{1 - x}$

---

18. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

19. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

20–25 Sketch the graph of a function that satisfies all of the given conditions.

20. $f'(x) > 0$ for all $x \neq 1$, vertical asymptote $x = 1$,
$f''(x) > 0$ if $x < 1$ or $x > 3$, $f''(x) < 0$ if $1 < x < 3$

21. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$

22. $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$,
$f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$,
$f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$

23. $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$,
$f'(-2) = 0$, $\lim_{x \to 2} |f'(x)| = \infty$, $f''(x) > 0$ if $x \neq 2$

24. $f(0) = f'(0) = f'(2) = f'(4) = f'(6) = 0$,
$f'(x) > 0$ if $0 < x < 2$ or $4 < x < 6$,
$f'(x) < 0$ if $2 < x < 4$ or $x > 6$,
$f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5$,
$f''(x) < 0$ if $1 < x < 3$ or $x > 5$, $f(-x) = f(x)$

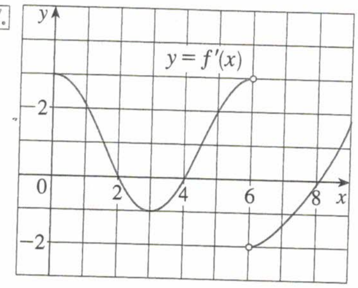
25. $f'(x) < 0$ and $f''(x) < 0$ for all x

---

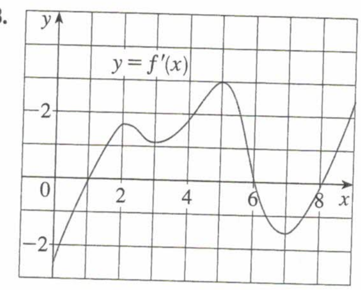
26. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
(a) Sketch a possible graph for f.
(b) How many solutions does the equation $f(x) = 0$ have? Why?
(c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

27–28 The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

27.



28.



29–40
(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the grapl
Check your work with a graphing device if you have one.

29. $f(x) = 2x^3 - 3x^2 - 12x$ **30.** $f(x) = 2 + 3x - x^3$

31. $f(x) = 2 + 2x^2 - x^4$ **32.** $g(x) = 200 + 8x^3 +$

33. $h(x) = (x + 1)^5 - 5x - 2$ **34.** $h(x) = x^5 - 2x^3 + x$

35. $A(x) = x\sqrt{x + 3}$ **36.** $B(x) = 3x^{2/3} - x$

37. $C(x) = x^{1/3}(x + 4)$ **38.** $G(x) = x - 4\sqrt{x}$

39. $f(\theta) = 2\cos\theta + \cos^2\theta$, $0 \leq \theta \leq 2\pi$

40. $f(t) = t + \cos t$, $-2\pi \leq t \leq 2\pi$

---

41. Suppose the derivative of a function f is $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$. On what interval is f increasing?

42. Use the methods of this section to sketch the curve $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ from each other?

43–44
(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.

(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

43. $f(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$

44. $f(x) = x + 2\cos x, \quad 0 \leqslant x \leqslant 2\pi$

45–46
(a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

45. $f(x) = \cos x + \frac{1}{2}\cos 2x, \quad 0 \leqslant x \leqslant 2\pi$

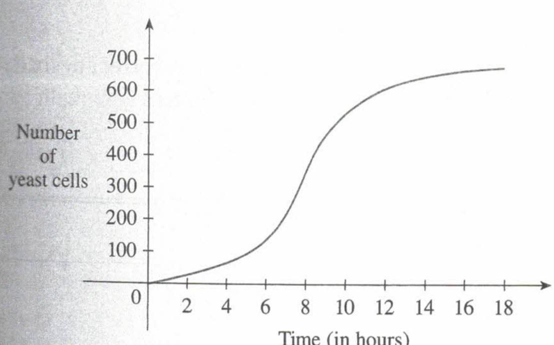
46. $f(x) = x^3(x - 2)^4$

47–48 Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph f''.

47. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$ **48.** $f(x) = \dfrac{(x + 1)^3(x^2 + 5)}{(x^3 + 1)(x^2 + 4)}$

---

49. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



50. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$ (b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$ (d) $f'(3) = -2, \quad f''(3) = -4$

51. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

52. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?

53. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at -2 and a local minimum value of 0 at 1.

54. Show that the curve $y = (1 + x)/(1 + x^2)$ has three points of inflection and they all lie on one straight line.

55. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

56–58 Assume that all of the functions are twice differentiable and the second derivatives are never 0.

56. (a) If f and g are concave upward on I, show that $f + g$ is concave upward on I.
(b) If f is positive and concave upward on I, show that the function $g(x) = [f(x)]^2$ is concave upward on I.

57. (a) If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.
(b) Show that part (a) remains true if f and g are both decreasing.
(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

58. Suppose f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

---

59. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

60. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}$$

61. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

62. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

63. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [*Hint:* Apply the First Derivative Test and Fermat's Theorem to the function $g = f'$.]

64. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

65. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

66. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

67. The three cases in the First Derivative Test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x} \qquad g(x) = x^4\left(2 + \sin\frac{1}{x}\right)$$

$$h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

4.4 LIMITS AT INFINITY; HORIZONTAL ASYMPTOTES

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

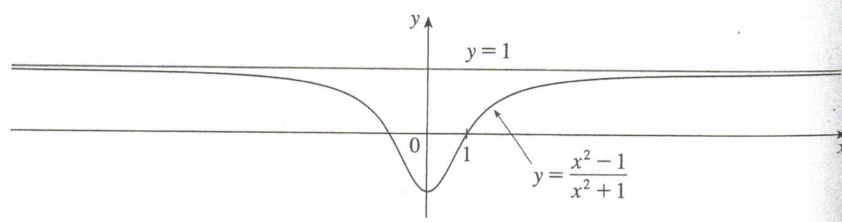


FIGURE 1

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

> **1 DEFINITION** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x \to \infty} f(x) = L$$
>
> means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim\limits_{x\to\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

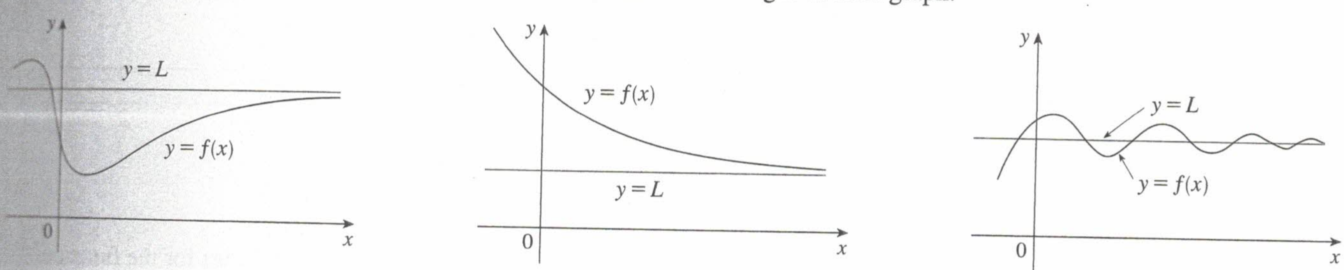


FIGURE 2
Examples illustrating $\lim\limits_{x\to\infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close as we like to 1. This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

> **2 DEFINITION** Let f be a function defined on some interval $(-\infty, a)$. Then
>
> $$\lim_{x \to -\infty} f(x) = L$$
>
> means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim\limits_{x\to-\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

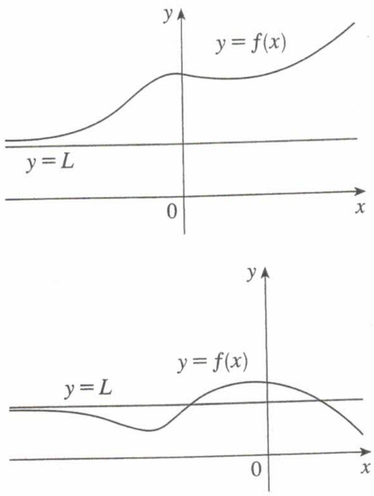


FIGURE 3
Examples illustrating $\lim_{x \to -\infty} f(x) = L$

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the li
we look to the far left of each graph.

3 DEFINITION The line $y = L$ is called a **horizontal asymptote** of the c
$y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horiz
tote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The curve $y = f(x)$ sketched in Figure 4 has both $y = -1$ and $y = 2$ as horiz
totes because

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

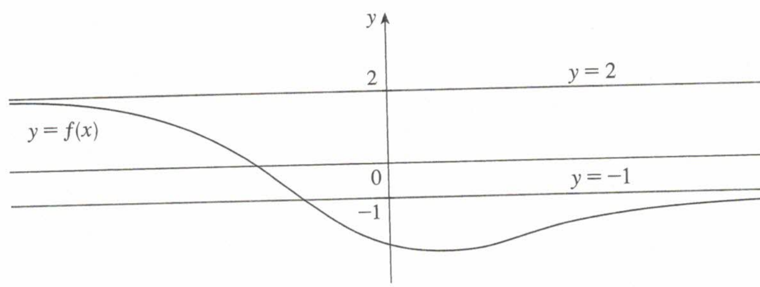


FIGURE 4

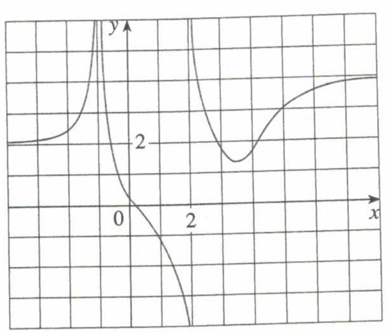


FIGURE 5

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for th
whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, bu
tive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreas
negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes.

EXAMPLE 2 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

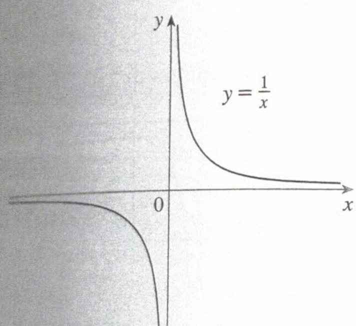


FIGURE 6

$\lim_{x\to\infty} \frac{1}{x} = 0, \ \lim_{x\to-\infty} \frac{1}{x} = 0$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.) □

Most of the Limit Laws that were given in Section 2.3 also hold for limits at infinity. It can be proved that the *Limit Laws listed in Section 2.3 (with the exception of Laws 9 and 10) are also valid if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$."* In particular, if we combine Laws 6 and 11 with the results of Example 2, we obtain the following important rule for calculating limits.

> **4 THEOREM** If $r > 0$ is a rational number, then
>
> $$\lim_{x\to\infty} \frac{1}{x^r} = 0$$
>
> If $r > 0$ is a rational number such that x^r is defined for all x, then
>
> $$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

V EXAMPLE 3 Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

and indicate which properties of limits are used at each stage.

SOLUTION As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator.

(We may assume that $x \neq 0$, since we are interested only in large values of x.) In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(3 - \dfrac{1}{x} - \dfrac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \dfrac{4}{x} + \dfrac{1}{x^2}\right)} \qquad \text{(by Limit Law 5)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \dfrac{1}{x} - 2\lim_{x\to\infty} \dfrac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \dfrac{1}{x} + \lim_{x\to\infty} \dfrac{1}{x^2}} \qquad \text{(by 1, 2, and 3)}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} \qquad \text{(by 7 and Theorem 4)}$$

$$= \frac{3}{5}$$

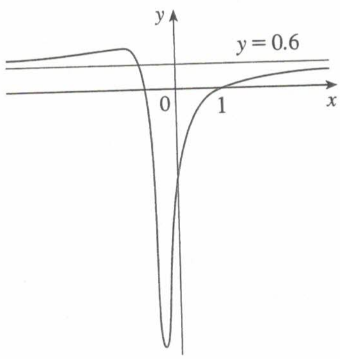


FIGURE 7

$y = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.

EXAMPLE 4 Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION Dividing both numerator and denominator by x and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x\to\infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} \qquad \text{(since } \sqrt{x^2} = x \text{ for } x > 0\text{)}$$

$$= \frac{\lim_{x\to\infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim_{x\to\infty}\left(3 - \dfrac{5}{x}\right)} = \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \dfrac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5\lim_{x\to\infty} \dfrac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for $x < 0$ we get

$$\frac{1}{x}\sqrt{2x^2 + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2 + 1} = -\sqrt{2 + \frac{1}{x^2}}$$

Therefore

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to-\infty}\frac{-\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}}=\frac{-\sqrt{2+\lim\limits_{x\to-\infty}\dfrac{1}{x^2}}}{3-5\lim\limits_{x\to-\infty}\dfrac{1}{x}}=-\frac{\sqrt{2}}{3}$$

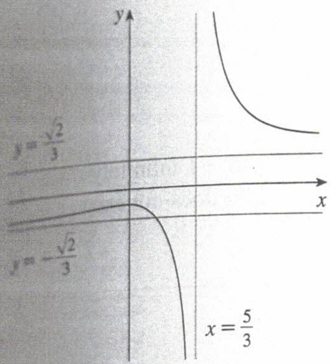


FIGURE 8
$y=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Thus the line $y=-\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x-5$, is 0, that is, when $x=\frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x>\frac{5}{3}$, then the denominator is close to 0 and $3x-5$ is positive. The numerator $\sqrt{2x^2+1}$ is always positive, so $f(x)$ is positive. Therefore

$$\lim_{x\to(5/3)^+}\frac{\sqrt{2x^2+1}}{3x-5}=\infty$$

If x is close to $\frac{5}{3}$ but $x<\frac{5}{3}$, then $3x-5<0$ and so $f(x)$ is large negative. Thus

$$\lim_{x\to(5/3)^-}\frac{\sqrt{2x^2+1}}{3x-5}=-\infty$$

The vertical asymptote is $x=\frac{5}{3}$. All three asymptotes are shown in Figure 8. □

EXAMPLE 5 Compute $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$.

SOLUTION Because both $\sqrt{x^2+1}$ and x are large when x is large, it's difficult to see what happens to their difference, so we use algebra to rewrite the function. We first multiply numerator and denominator by the conjugate radical:

■ We can think of the given function as having a denominator of 1.

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to\infty}\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

The Squeeze Theorem could be used to show that this limit is 0. But an easier method is to divide numerator and denominator by x. Doing this and using the Limit Laws, we obtain

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{\dfrac{\sqrt{x^2+1}+x}{x}}$$

$$=\lim_{x\to\infty}\frac{\dfrac{1}{x}}{\sqrt{1+\dfrac{1}{x^2}}+1}=\frac{0}{\sqrt{1+0}+1}=0$$

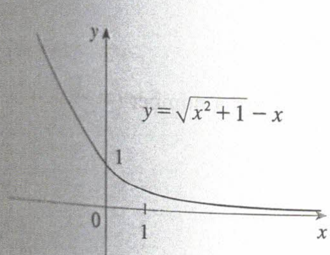


FIGURE 9

Figure 9 illustrates this result. □

EXAMPLE 6 Evaluate $\lim_{x \to \infty} \sin \frac{1}{x}$.

■ The problem-solving strategy for Example 6 is *introducing something extra* (see page 54). Here, the something extra, the auxiliary aid, is the new variable t.

SOLUTION If we let $t = 1/x$, then $t \to 0^+$ as $x \to \infty$. Therefore

$$\lim_{x \to \infty} \sin \frac{1}{x} = \lim_{t \to 0^+} \sin t = 0$$

(See Exercise 69.)

EXAMPLE 7 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x \to \infty} \sin x$ does not exist.

INFINITE LIMITS AT INFINITY

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

EXAMPLE 8 Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by taking x large enough. Therefore we can write

$$\lim_{x \to \infty} x^3 = \infty$$

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

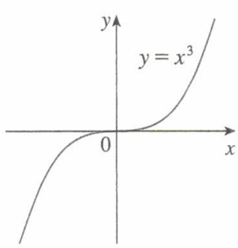


FIGURE 10
$\lim_{x \to \infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 10.

EXAMPLE 9 Find $\lim_{x \to \infty} (x^2 - x)$.

SOLUTION It would be **wrong** to write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 - \lim_{x \to \infty} x = \infty - \infty$$

The Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too.

EXAMPLE 10 Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is just x:

$$\lim_{x\to\infty} \frac{x^2 + x}{3 - x} = \lim_{x\to\infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to -1$ as $x \to \infty$. □

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without computing derivatives.

EXAMPLE 11 Sketch the graph of $y = (x - 2)^4(x + 1)^3(x - 1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The y-intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x-intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x - 2)^4$ is positive, the function doesn't change sign at 2; thus the graph doesn't cross the x-axis at 2. The graph crosses the axis at -1 and 1.

When x is large positive, all three factors are large, so

$$\lim_{x\to\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x\to-\infty} (x - 2)^4(x + 1)^3(x - 1) = \infty$$

Combining this information, we give a rough sketch of the graph in Figure 11. □

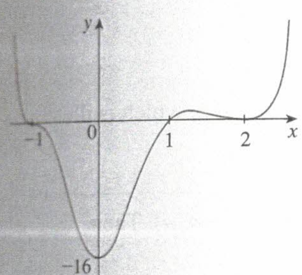


FIGURE 11
$y = (x - 2)^4(x + 1)^3(x - 1)$

PRECISE DEFINITIONS

Definition 1 can be stated precisely as follows.

> **5 DEFINITION** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x\to\infty} f(x) = L$$
>
> means that for every $\varepsilon > 0$ there is a corresponding number N such that
>
> $$\text{if} \quad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where N depends on ε). Graphically it says that by choosing x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines

$y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 12. This must be true no matter how small we choose ε. Figure 13 shows that if a smaller value of ε is chosen, then a larger value of N may be required.

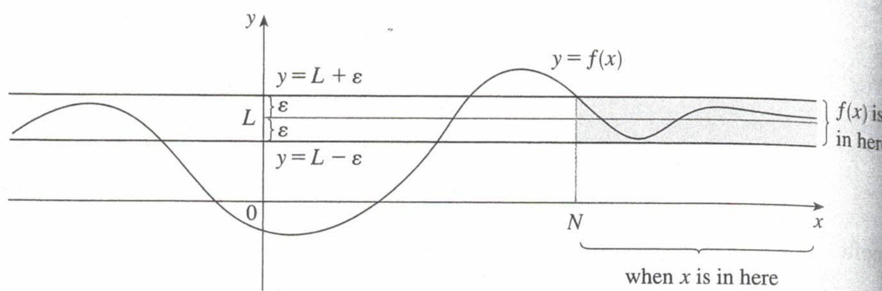


FIGURE 12
$\lim_{x\to\infty} f(x) = L$

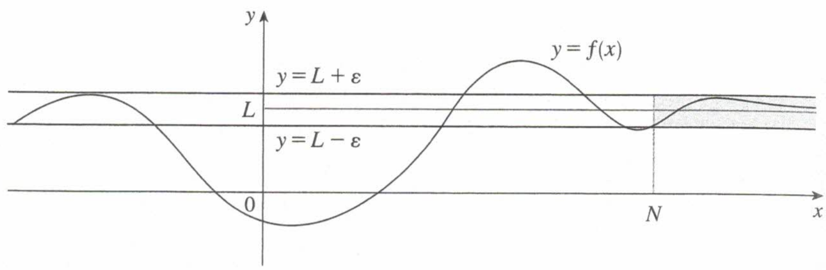


FIGURE 13
$\lim_{x\to\infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 6, which is illustrated in Figure 14.

> **6 DEFINITION** Let f be a function defined on some interval $(-\infty, a)$. Then
>
> $$\lim_{x\to-\infty} f(x) = L$$
>
> means that for every $\varepsilon > 0$ there is a corresponding number N such that
>
> $$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

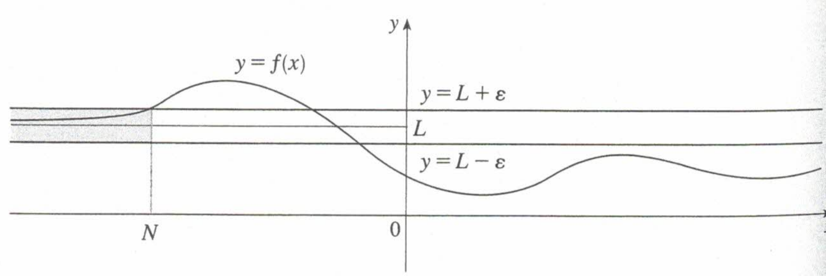


FIGURE 14
$\lim_{x\to-\infty} f(x) = L$

In Example 3 we calculated that

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a graphing device to relate this statement to Definition 5 with $L = \frac{3}{5}$ and $\varepsilon = 0.1$.

TEC In Module 2.4/4.4 you can explore the precise definition of a limit both graphically and numerically.

EXAMPLE 12 Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 15. Then we use the cursor to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding to be safe, we can say that

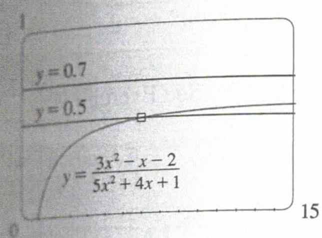


FIGURE 15

$$\text{if} \quad x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 5. □

EXAMPLE 13 Use Definition 5 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

SOLUTION Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if} \quad x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 5,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 16 illustrates the proof by showing some values of ε and the corresponding values of N.

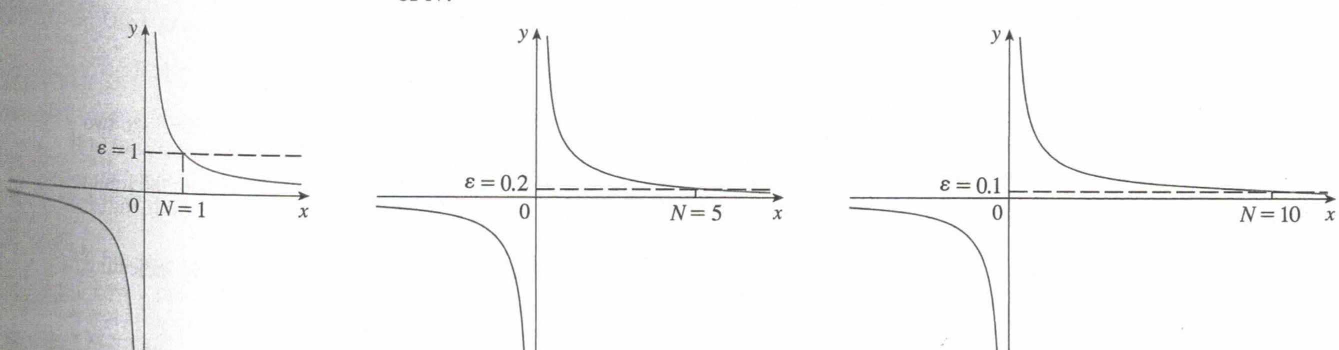


FIGURE 16

□

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.

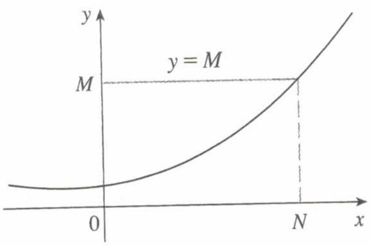


FIGURE 17
$\lim_{x\to\infty} f(x) = \infty$

7 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

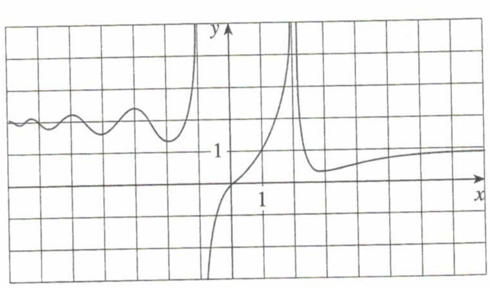
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 70.)

4.4 EXERCISES

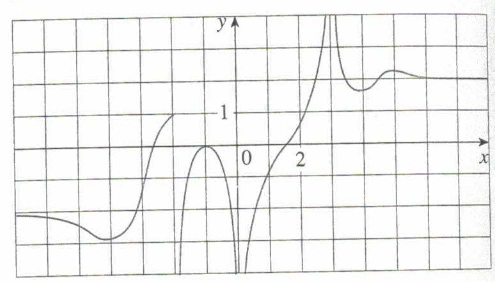
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x\to\infty} f(x) = 5$ (b) $\lim_{x\to-\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x\to 2} f(x)$ (b) $\lim_{x\to -1^-} f(x)$
(c) $\lim_{x\to -1^+} f(x)$ (d) $\lim_{x\to\infty} f(x)$
(e) $\lim_{x\to -\infty} f(x)$ (f) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x\to\infty} g(x)$ (b) $\lim_{x\to-\infty} g(x)$
(c) $\lim_{x\to 3} g(x)$ (d) $\lim_{x\to 0} g(x)$
(e) $\lim_{x\to -2^+} g(x)$ (f) The equations of the asymptotes



5. Guess the value of the limit

$$\lim_{x\to\infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

6. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x\to\infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

7–8 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

7. $\lim_{x\to\infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

8. $\lim_{x\to\infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

9–30 Find the limit.

9. $\lim_{x\to\infty} \dfrac{1}{2x+3}$

10. $\lim_{x\to\infty} \dfrac{3x+5}{x-4}$

11. $\lim_{x\to-\infty} \dfrac{1-x-x^2}{2x^2-7}$

12. $\lim_{y\to\infty} \dfrac{2-3y^2}{5y^2+4y}$

13. $\lim_{x\to\infty} \dfrac{x^3+5x}{2x^3-x^2+4}$

14. $\lim_{t\to-\infty} \dfrac{t^2+2}{t^3+t^2-1}$

15. $\lim_{u\to\infty} \dfrac{4u^4+5}{(u^2-2)(2u^2-1)}$

16. $\lim_{x\to\infty} \dfrac{x+2}{\sqrt{9x^2+1}}$

17. $\lim_{x\to\infty} \dfrac{\sqrt{9x^6-x}}{x^3+1}$

18. $\lim_{x\to-\infty} \dfrac{\sqrt{9x^6-x}}{x^3+1}$

19. $\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)$

20. $\lim_{x\to-\infty} \left(x+\sqrt{x^2+2x}\right)$

21. $\lim_{x\to\infty} \left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

22. $\lim_{x\to\infty} \cos x$

23. $\lim_{x\to\infty} \dfrac{x+x^3+x^5}{1-x^2+x^4}$

24. $\lim_{x\to\infty} \sqrt{x^2+1}$

25. $\lim_{x\to-\infty} (x^4+x^5)$

26. $\lim_{x\to\infty} \dfrac{x^3-2x+3}{5-2x^2}$

27. $\lim_{x\to\infty} \left(x-\sqrt{x}\right)$

28. $\lim_{x\to\infty} (x^2-x^4)$

29. $\lim_{x\to\infty} x\sin\dfrac{1}{x}$

30. $\lim_{x\to\infty} \sqrt{x}\sin\dfrac{1}{x}$

31. (a) Estimate the value of

$$\lim_{x\to-\infty} \left(\sqrt{x^2+x+1}+x\right)$$

by graphing the function $f(x)=\sqrt{x^2+x+1}+x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

32. (a) Use a graph of

$$f(x)=\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}$$

to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

33–38 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

33. $y=\dfrac{2x+1}{x-2}$

34. $y=\dfrac{x^2+1}{2x^2-3x-2}$

35. $y=\dfrac{2x^2+x-1}{x^2+x-2}$

36. $y=\dfrac{1+x^4}{x^2-x^4}$

37. $y=\dfrac{x^3-x}{x^2-6x+5}$

38. $F(x)=\dfrac{x-9}{\sqrt{4x^2+3x+2}}$

39. Estimate the horizontal asymptote of the function

$$f(x)=\frac{3x^3+500x^2}{x^3+500x^2+100x+2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

40. (a) Graph the function

$$f(x)=\frac{\sqrt{2x^2+1}}{3x-5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} \quad \text{and} \quad \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

41. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x\to\pm\infty} f(x)=0, \quad \lim_{x\to 0} f(x)=-\infty, \quad f(2)=0,$$

$$\lim_{x\to 3^-} f(x)=\infty, \quad \lim_{x\to 3^+} f(x)=-\infty$$

42. Find a formula for a function that has vertical asymptotes $x=1$ and $x=3$ and horizontal asymptote $y=1$.

43–46 Find the horizontal asymptotes of the curve and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

43. $y=\dfrac{1-x}{1+x}$

44. $y=\dfrac{1+2x^2}{1+x^2}$

45. $y=\dfrac{x}{x^2+1}$

46. $y=\dfrac{x}{\sqrt{x^2+1}}$

47–50 Find the limits as $x\to\infty$ and as $x\to-\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 11.

47. $y=x^4-x^6$

48. $y=x^3(x+2)^2(x-1)$

49. $y = (3 - x)(1 + x)^2(1 - x)^4$

50. $y = x^2(x^2 - 1)^2(x + 2)$

51–54 Sketch the graph of a function that satisfies all of the given conditions.

51. $f'(2) = 0, \quad f(2) = -1, \quad f(0) = 0,$
$f'(x) < 0$ if $0 < x < 2$, $\quad f'(x) > 0$ if $x > 2$,
$f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4$,
$f''(x) > 0$ if $1 < x < 4$, $\quad \lim_{x\to\infty} f(x) = 1$,
$f(-x) = f(x)$ for all x

52. $f'(2) = 0, \quad f'(0) = 1, \quad f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $x > 2$, $\quad f''(x) < 0$ if $0 < x < 4$,
$f''(x) > 0$ if $x > 4$, $\quad \lim_{x\to\infty} f(x) = 0$,
$f(-x) = -f(x)$ for all x

53. $f(1) = f'(1) = 0, \quad \lim_{x\to 2^+} f(x) = \infty, \quad \lim_{x\to 2^-} f(x) = -\infty,$
$\lim_{x\to 0} f(x) = -\infty, \quad \lim_{x\to -\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 0,$
$f''(x) > 0$ for $x > 2$, $\quad f''(x) < 0$ for $x < 0$ and for $0 < x < 2$

54. $g(0) = 0, \quad g''(x) < 0$ for $x \neq 0$, $\quad \lim_{x\to -\infty} g(x) = \infty,$
$\lim_{x\to\infty} g(x) = -\infty, \quad \lim_{x\to 0^-} g'(x) = -\infty,$
$\lim_{x\to 0^+} g'(x) = \infty$

---

55. (a) Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

56. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

57. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

58. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\displaystyle\lim_{x\to 0^+} x^n$ (b) $\displaystyle\lim_{x\to 0^-} x^n$
(c) $\displaystyle\lim_{x\to\infty} x^n$ (d) $\displaystyle\lim_{x\to -\infty} x^n$

59. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

60. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

61. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

62. For the limit

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 5 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

63. For the limit

$$\lim_{x\to -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 6 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

64. For the limit

$$\lim_{x\to\infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 7 by finding a value of N that corresponds to $M = 100$.

65. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 4, we have the statement

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 5.

66. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
(b) Taking $r = \frac{1}{2}$ in Theorem 4, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 5.

67. Use Definition 6 to prove that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

68. Prove, using Definition 7, that $\lim_{x \to \infty} x^3 = \infty$.

69. Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

if these limits exist.

70. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

4.5 SUMMARY OF CURVE SKETCHING

So far we have been concerned with some particular aspects of curve sketching: domain, range, and symmetry in Chapter 1; limits, continuity, and vertical asymptotes in Chapter 2; derivatives and tangents in Chapter 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and horizontal asymptotes in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You might ask: Why don't we just use a graphing calculator or computer to graph a curve? Why do we need to use calculus?

It's true that modern technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. We saw in Section 1.4 that it is extremely important to choose an appropriate viewing rectangle to avoid getting a misleading graph. (See especially Examples 1, 3, 4, and 5 in that section.) The use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points *exactly* instead of approximately.

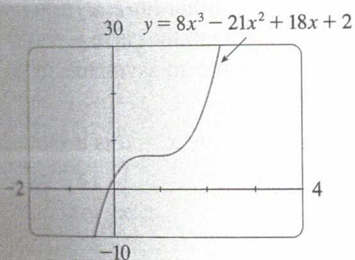


FIGURE 1

For instance, Figure 1 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: It has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 2. Without calculus, we could easily have overlooked it.

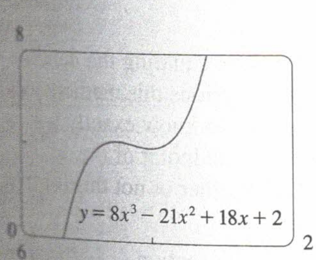


FIGURE 2

In the next section we will graph functions by using the interaction between calculus and graphing devices. In this section we draw graphs by first considering the following information. We don't assume that you have a graphing device, but if you do have one you should use it as a check on your work.

GUIDELINES FOR SKETCHING A CURVE

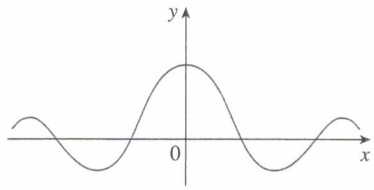
The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

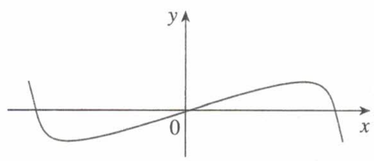
B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 3

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).

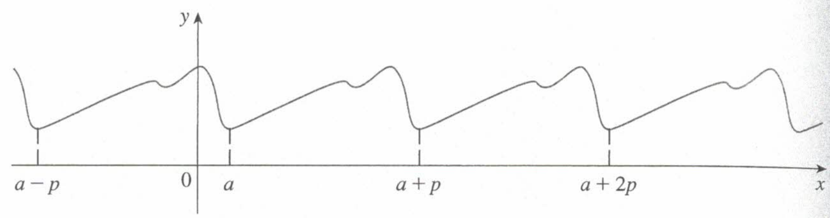


FIGURE 4
Periodic function: translational symmetry

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 4.4 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

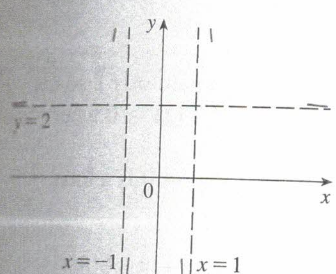


FIGURE 5
Preliminary sketch

We have shown the curve approaching its horizontal asymptote from above in Figure 5. This is confirmed by the intervals of increase and decrease.

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

E.

$$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ $(x \neq -1)$ and $f'(x) < 0$ when $x > 0$ $(x \neq 1)$, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

G.

$$f''(x) = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

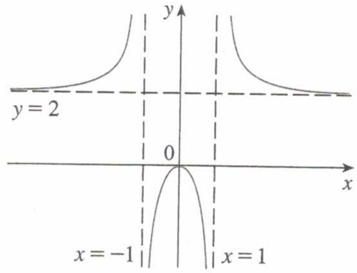


FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure 6. □

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x + 1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x + 1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x + 1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E.
$$f'(x) = \frac{2x\sqrt{x + 1} - x^2 \cdot 1/(2\sqrt{x + 1})}{x + 1} = \frac{x(3x + 4)}{2(x + 1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

G.
$$f''(x) = \frac{2(x + 1)^{3/2}(6x + 4) - (3x^2 + 4x)3(x + 1)^{1/2}}{4(x + 1)^3} = \frac{3x^2 + 8x + 8}{4(x + 1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

H. The curve is sketched in Figure 7. □

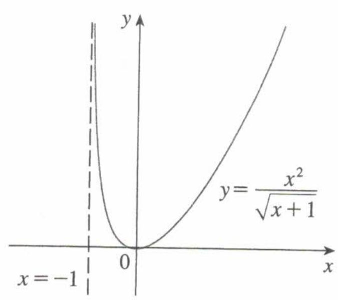


FIGURE 7

EXAMPLE 3 Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. The domain is $\mathbb{R}$.

B. The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (2n + 1)\pi/2$, where n is an integer.

C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.

D. Asymptotes: None

E. $$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

Thus $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.

F. From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.

G. If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.

H. The graph of the function restricted to $0 \leqslant x \leqslant 2\pi$ is shown in Figure 8. Then we extend it, using periodicity, to the complete graph in Figure 9.

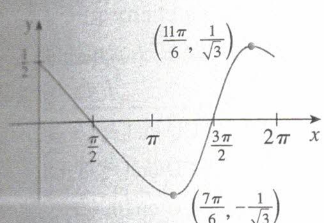


FIGURE 8

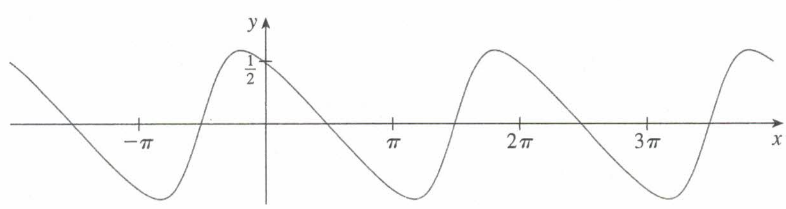


FIGURE 9

□

SLANT ASYMPTOTES

Some curves have asymptotes that are *oblique,* that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 10. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

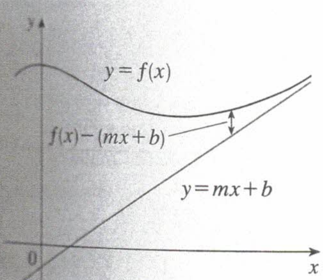


FIGURE 10

V EXAMPLE 4 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$
$f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long divis

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E.
$$f'(x) = \frac{3x^2(x^2 + 1) - x^3 \cdot 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local max minimum.

G.
$$f''(x) = \frac{(4x^3 + 6x)(x^2 + 1)^2 - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{2x(3 -}{(x^2 +}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0,$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{}$

The points of inflection are $(-\sqrt{3}, -\frac{3}{4}\sqrt{3})$, $(0, 0)$, and $(\sqrt{3}, \frac{3}{4}\sqrt{3})$.

H. The graph of f is sketched in Figure 11.

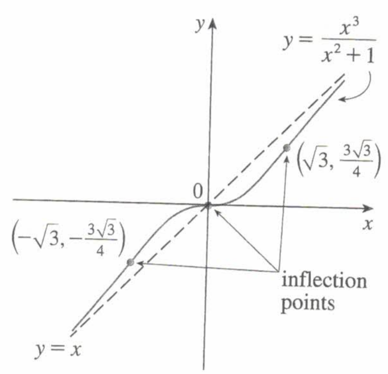


FIGURE 11

4.5 EXERCISES

1–38 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + x$

2. $y = x^3 + 6x^2 + 9x$

3. $y = 2 - 15x + 9x^2 - x^3$

4. $y = 8x^2 - x^4$

5. $y = x^4 + 4x^3$

6. $y = x(x + 2)^3$

7. $y = 2x^5 - 5x^2 + 1$

8. $y = (4 - x^2)^5$

9. $y = \frac{x}{x - 1}$

10. $y = \frac{x^2 - 4}{x^2 - 2x}$

11. $y = \frac{1}{x^2 - 9}$

12. $y = \frac{x}{x^2 - 9}$

13. $y = \frac{x}{x^2 + 9}$

14. $y = \frac{x^2}{x^2 + 9}$

15. $y = \frac{x - 1}{x^2}$

16. $y = 1 + \frac{1}{x} + \frac{1}{x^2}$

17. $y = \frac{x^2}{x^2 + 3}$

18. $y = \frac{x}{x^3 - 1}$

19. $y = x\sqrt{5 - x}$

20. $y = 2\sqrt{x} - x$

21. $y = \sqrt{x^2 + x - 2}$

22. $y = \sqrt{x^2 + x} - x$

23. $y = \frac{x}{\sqrt{x^2 + 1}}$

24. $y = x\sqrt{2 - x^2}$

25. $y = \dfrac{\sqrt{1 - x^2}}{x}$

26. $y = \dfrac{x}{\sqrt{x^2 - 1}}$

27. $y = x - 3x^{1/3}$

28. $y = x^{5/3} - 5x^{2/3}$

29. $y = \sqrt[3]{x^2 - 1}$

30. $y = \sqrt[3]{x^3 + 1}$

31. $y = 3\sin x - \sin^3 x$

32. $y = x + \cos x$

33. $y = x\tan x, \quad -\pi/2 < x < \pi/2$

34. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$

35. $y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi$

36. $y = \sec x + \tan x, \quad 0 < x < \pi/2$

37. $y = \dfrac{\sin x}{1 + \cos x}$

38. $y = \dfrac{\sin x}{2 + \cos x}$

39. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

40. In the theory of relativity, the energy of a particle is

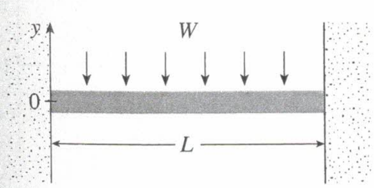
$$E = \sqrt{m_0^2 c^4 + h^2 c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

41. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

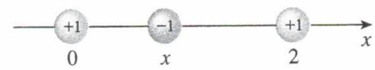
where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



42. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x-2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?



43–46 Find an equation of the slant asymptote. Do not sketch the curve.

43. $y = \dfrac{x^2 + 1}{x + 1}$

44. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

45. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$

46. $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$

47–52 Use the guidelines of this section to sketch the curve. In guideline D find an equation of the slant asymptote.

47. $y = \dfrac{-2x^2 + 5x - 1}{2x - 1}$

48. $y = \dfrac{x^2 + 12}{x - 2}$

49. $xy = x^2 + 4$

50. $xy = x^2 + x + 1$

51. $y = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$

52. $y = \dfrac{(x + 1)^3}{(x - 1)^2}$

53. Show that the curve $y = \sqrt{4x^2 + 9}$ has two slant asymptotes: $y = 2x$ and $y = -2x$. Use this fact to help sketch the curve.

54. Show that the curve $y = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$. Use this fact to help sketch the curve.

55. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

56. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

57. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 56. Then use your results to help sketch the graph of f.

58. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

4.6 GRAPHING WITH CALCULUS *AND* CALCULATORS

If you have not already read Section 1.4, you should do so now. In particular, it explains how to avoid some of the pitfalls of graphing devices by choosing appropriate viewing rectangles.

The method we used to sketch curves in the preceding section was a culmination of much of our study of differential calculus. The graph was the final object that we produced. In this section our point of view is completely different. Here we *start* with a graph produced by a graphing calculator or computer and then we refine it. We use calculus to make sure that we reveal all the important aspects of the curve. And with the use of graphing devices we can tackle curves that would be far too complicated to consider without technology. The theme is the *interaction* between calculus and calculators.

EXAMPLE 1 Graph the polynomial $f(x) = 2x^6 + 3x^5 + 3x^3 - 2x^2$. Use the graphs of f' and f'' to estimate all maximum and minimum points and intervals of concavity.

SOLUTION If we specify a domain but not a range, many graphing devices will deduce a suitable range from the values computed. Figure 1 shows the plot from one such device if we specify that $-5 \leq x \leq 5$. Although this viewing rectangle is useful for showing that the asymptotic behavior (or end behavior) is the same as for $y = 2x^6$, it is obviously hiding some finer detail. So we change to the viewing rectangle $[-3, 2]$ by $[-50, 100]$ shown in Figure 2.

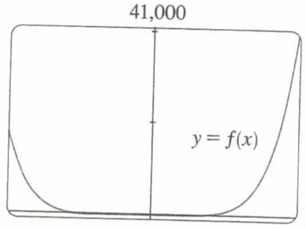


FIGURE 1

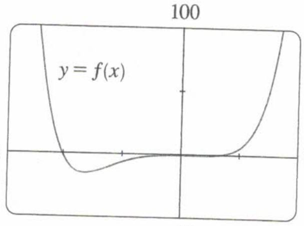


FIGURE 2

From this graph it appears that there is an absolute minimum value of about -15.33 when $x \approx -1.62$ (by using the cursor) and f is decreasing on $(-\infty, -1.62)$ and increasing on $(-1.62, \infty)$. Also, there appears to be a horizontal tangent at the origin and inflection points when $x = 0$ and when x is somewhere between -2 and -1.

Now let's try to confirm these impressions using calculus. We differentiate and get

$$f'(x) = 12x^5 + 15x^4 + 9x^2 - 4x$$

$$f''(x) = 60x^4 + 60x^3 + 18x - 4$$

When we graph f' in Figure 3 we see that $f'(x)$ changes from negative to positive when $x \approx -1.62$; this confirms (by the First Derivative Test) the minimum value that we found earlier. But, perhaps to our surprise, we also notice that $f'(x)$ changes from positive to negative when $x = 0$ and from negative to positive when $x \approx 0.35$. This means that f has a local maximum at 0 and a local minimum when $x \approx 0.35$, but these were hidden in Figure 2. Indeed, if we now zoom in toward the origin in Figure 4, we see what we missed before: a local maximum value of 0 when $x = 0$ and a local minimum value of about -0.1 when $x \approx 0.35$.

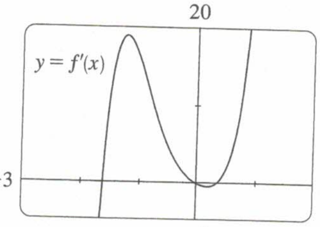


FIGURE 3

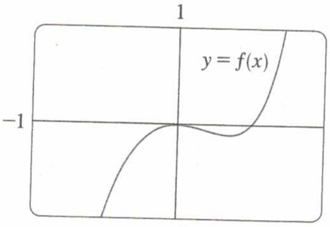


FIGURE 4

What about concavity and inflection points? From Figures 2 and 4 there appear to be inflection points when x is a little to the left of -1 and when x is a little to the right of 0. But it's difficult to determine inflection points from the graph of f, so we graph the sec-

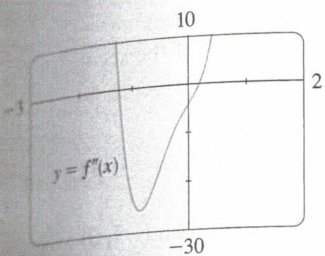


FIGURE 5

ond derivative f'' in Figure 5. We see that f'' changes from positive to negative when $x \approx -1.23$ and from negative to positive when $x \approx 0.19$. So, correct to two decimal places, f is concave upward on $(-\infty, -1.23)$ and $(0.19, \infty)$ and concave downward on $(-1.23, 0.19)$. The inflection points are $(-1.23, -10.18)$ and $(0.19, -0.05)$.

We have discovered that no single graph reveals all the important features of this polynomial. But Figures 2 and 4, when taken together, do provide an accurate picture. □

EXAMPLE 2 Draw the graph of the function

$$f(x) = \frac{x^2 + 7x + 3}{x^2}$$

in a viewing rectangle that contains all the important features of the function. Estimate the maximum and minimum values and the intervals of concavity. Then use calculus to find these quantities exactly.

SOLUTION Figure 6, produced by a computer with automatic scaling, is a disaster. Some graphing calculators use $[-10, 10]$ by $[-10, 10]$ as the default viewing rectangle, so let's try it. We get the graph shown in Figure 7; it's a major improvement.

The y-axis appears to be a vertical asymptote and indeed it is because

$$\lim_{x \to 0} \frac{x^2 + 7x + 3}{x^2} = \infty$$

Figure 7 also allows us to estimate the x-intercepts: about -0.5 and -6.5. The exact values are obtained by using the quadratic formula to solve the equation $x^2 + 7x + 3 = 0$; we get $x = \left(-7 \pm \sqrt{37}\right)/2$.

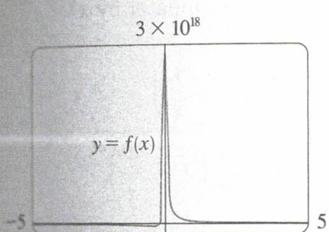


FIGURE 6

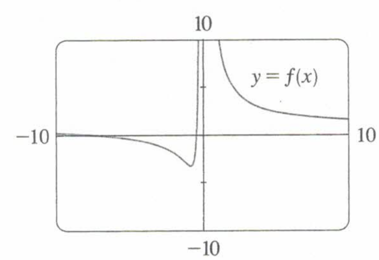


FIGURE 7

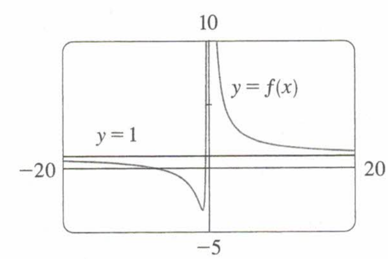


FIGURE 8

To get a better look at horizontal asymptotes, we change to the viewing rectangle $[-20, 20]$ by $[-5, 10]$ in Figure 8. It appears that $y = 1$ is the horizontal asymptote and this is easily confirmed:

$$\lim_{x \to \pm\infty} \frac{x^2 + 7x + 3}{x^2} = \lim_{x \to \pm\infty} \left(1 + \frac{7}{x} + \frac{3}{x^2}\right) = 1$$

To estimate the minimum value we zoom in to the viewing rectangle $[-3, 0]$ by $[-4, 2]$ in Figure 9. The cursor indicates that the absolute minimum value is about -3.1 when $x \approx -0.9$, and we see that the function decreases on $(-\infty, -0.9)$ and $(0, \infty)$ and increases on $(-0.9, 0)$. The exact values are obtained by differentiating:

$$f'(x) = -\frac{7}{x^2} - \frac{6}{x^3} = -\frac{7x + 6}{x^3}$$

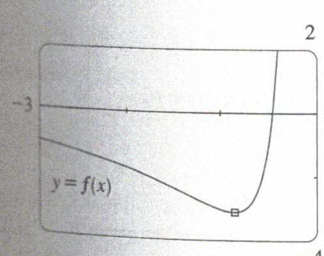


FIGURE 9

This shows that $f'(x) > 0$ when $-\frac{6}{7} < x < 0$ and $f'(x) < 0$ when $x < -\frac{6}{7}$ and when $x > 0$. The exact minimum value is $f\left(-\frac{6}{7}\right) = -\frac{37}{12} \approx -3.08$.

Figure 9 also shows that an inflection point occurs somewhere between $x = -1$ and $x = -2$. We could estimate it much more accurately using the graph of the second derivative, but in this case it's just as easy to find exact values. Since

$$f''(x) = \frac{14}{x^3} + \frac{18}{x^4} = \frac{2(7x + 9)}{x^4}$$

we see that $f''(x) > 0$ when $x > -\frac{9}{7}$ $(x \neq 0)$. So f is concave upward on $\left(-\frac{9}{7}, 0\right)$ and $(0, \infty)$ and concave downward on $\left(-\infty, -\frac{9}{7}\right)$. The inflection point is $\left(-\frac{9}{7}, -\frac{71}{27}\right)$.

The analysis using the first two derivatives shows that Figures 7 and 8 display all the major aspects of the curve. □

V EXAMPLE 3 Graph the function $f(x) = \dfrac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4}$.

SOLUTION Drawing on our experience with a rational function in Example 2, let's start by graphing f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$. From Figure 10 we have the feeling that we are going to have to zoom in to see some finer detail and also zoom out to see the larger picture. But, as a guide to intelligent zooming, let's first take a close look at the expression for $f(x)$. Because of the factors $(x - 2)^2$ and $(x - 4)^4$ in the denominator, we expect $x = 2$ and $x = 4$ to be the vertical asymptotes. Indeed

$$\lim_{x \to 2} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty \qquad \text{and} \qquad \lim_{x \to 4} \frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \infty$$

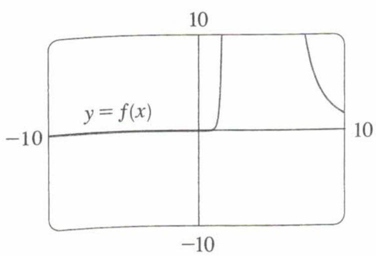


FIGURE 10

To find the horizontal asymptotes, we divide numerator and denominator by x^6:

$$\frac{x^2(x + 1)^3}{(x - 2)^2(x - 4)^4} = \frac{\dfrac{x^2}{x^3} \cdot \dfrac{(x + 1)^3}{x^3}}{\dfrac{(x - 2)^2}{x^2} \cdot \dfrac{(x - 4)^4}{x^4}} = \frac{\dfrac{1}{x}\left(1 + \dfrac{1}{x}\right)^3}{\left(1 - \dfrac{2}{x}\right)^2\left(1 - \dfrac{4}{x}\right)^4}$$

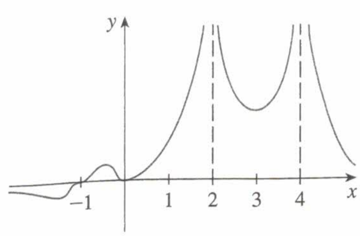


FIGURE 11

This shows that $f(x) \to 0$ as $x \to \pm\infty$, so the x-axis is a horizontal asymptote.

It is also very useful to consider the behavior of the graph near the x-intercepts using an analysis like that in Example 11 in Section 4.4. Since x^2 is positive, $f(x)$ does not change sign at 0 and so its graph doesn't cross the x-axis at 0. But, because of the factor $(x + 1)^3$, the graph does cross the x-axis at -1 and has a horizontal tangent there. Putting all this information together, but without using derivatives, we see that the curve has to look something like the one in Figure 11.

Now that we know what to look for, we zoom in (several times) to produce the graphs in Figures 12 and 13 and zoom out (several times) to get Figure 14.

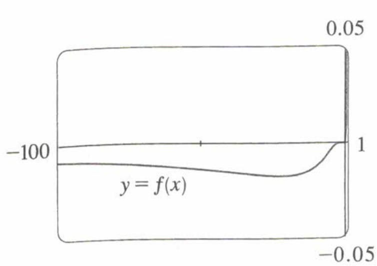


FIGURE 12

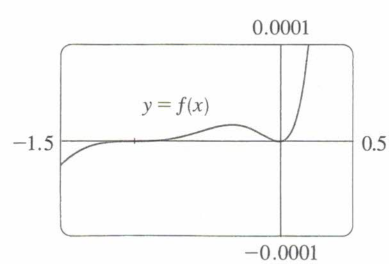


FIGURE 13

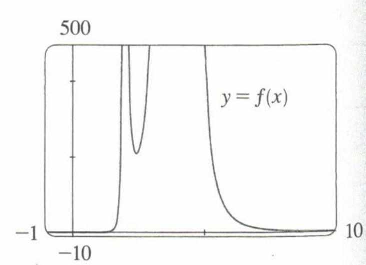


FIGURE 14

We can read from these graphs that the absolute minimum is about -0.02 and occurs when $x \approx -20$. There is also a local maximum ≈ 0.00002 when $x \approx -0.3$ and a local minimum ≈ 211 when $x \approx 2.5$. These graphs also show three inflection points near -35, -5, and -1 and two between -1 and 0. To estimate the inflection points closely we would need to graph f'', but to compute f'' by hand is an unreasonable chore. If you have a computer algebra system, then it's easy to do (see Exercise 13).

We have seen that, for this particular function, *three* graphs (Figures 12, 13, and 14) are necessary to convey all the useful information. The only way to display all these features of the function on a single graph is to draw it by hand. Despite the exaggerations and distortions, Figure 11 does manage to summarize the essential nature of the function. □

The family of functions

$$f(x) = \sin(x + \sin cx)$$

where c is a constant, occurs in applications to frequency modulation (FM) synthesis. A sine wave is modulated by a wave with a different frequency ($\sin cx$). The case where $c = 2$ is studied in Example 4. Exercise 19 explores another special case.

EXAMPLE 4 Graph the function $f(x) = \sin(x + \sin 2x)$. For $0 \leq x \leq \pi$, estimate all maximum and minimum values, intervals of increase and decrease, and inflection points correct to one decimal place.

SOLUTION We first note that f is periodic with period 2π. Also, f is odd and $|f(x)| \leq 1$ for all x. So the choice of a viewing rectangle is not a problem for this function: We start with $[0, \pi]$ by $[-1.1, 1.1]$. (See Figure 15.)

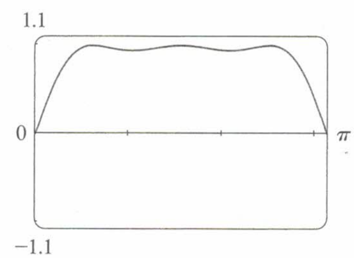


FIGURE 15

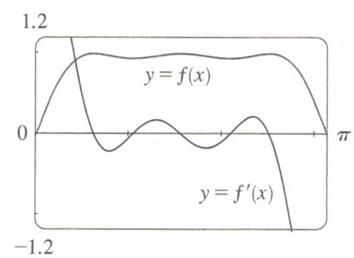


FIGURE 16

It appears that there are three local maximum values and two local minimum values in that window. To confirm this and locate them more accurately, we calculate that

$$f'(x) = \cos(x + \sin 2x) \cdot (1 + 2\cos 2x)$$

and graph both f and f' in Figure 16.

Using zoom-in and the First Derivative Test, we find the following values to one decimal place.

Intervals of increase:	$(0, 0.6)$, $(1.0, 1.6)$, $(2.1, 2.5)$
Intervals of decrease:	$(0.6, 1.0)$, $(1.6, 2.1)$, $(2.5, \pi)$
Local maximum values:	$f(0.6) \approx 1$, $f(1.6) \approx 1$, $f(2.5) \approx 1$
Local minimum values:	$f(1.0) \approx 0.94$, $f(2.1) \approx 0.94$

The second derivative is

$$f''(x) = -(1 + 2\cos 2x)^2 \sin(x + \sin 2x) - 4 \sin 2x \cos(x + \sin 2x)$$

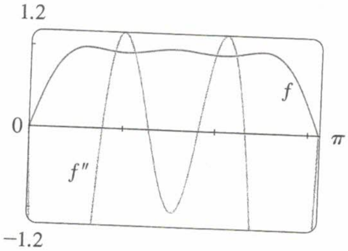


FIGURE 17

Graphing both f and f'' in Figure 17, we obtain the following approximate v

Concave upward on:	$(0.8, 1.3)$, $(1.8, 2.3)$
Concave downward on:	$(0, 0.8)$, $(1.3, 1.8)$, $(2.3, \pi)$
Inflection points:	$(0, 0)$, $(0.8, 0.97)$, $(1.3, 0.97)$, $(1.8, 0.97)$, $(2.3$

Having checked that Figure 15 does indeed represent f accurately for $0 \leqslant$ we can state that the extended graph in Figure 18 represents f accurately for $-2\pi \leqslant x \leqslant 2\pi$.

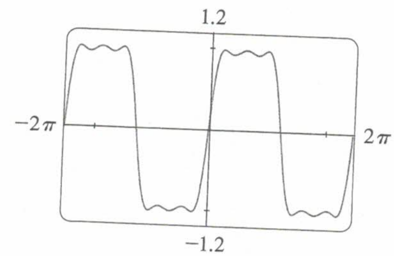


FIGURE 18

Our final example is concerned with *families* of functions. As discussed in S this means that the functions in the family are related to each other by a formul tains one or more arbitrary constants. Each value of the constant gives rise to a the family and the idea is to see how the graph of the function changes as th changes.

EXAMPLE 5 How does the graph of $f(x) = 1/(x^2 + 2x + c)$ vary as c varies

SOLUTION The graphs in Figures 19 and 20 (the special cases $c = 2$ and $c = -2$) two very different-looking curves. Before drawing any more graphs, let's see wh bers of this family have in common. Since

$$\lim_{x \to \pm\infty} \frac{1}{x^2 + 2x + c} = 0$$

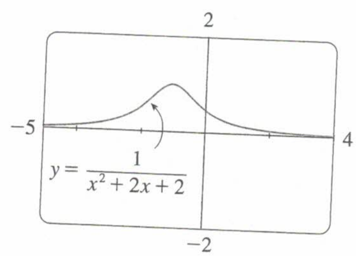


FIGURE 19
$c = 2$

for any value of c, they all have the x-axis as a horizontal asymptote. A vertical a tote will occur when $x^2 + 2x + c = 0$. Solving this quadratic equation, we get $x = -1 \pm \sqrt{1 - c}$. When $c > 1$, there is no vertical asymptote (as in Figure 19 $c = 1$, the graph has a single vertical asymptote $x = -1$ because

$$\lim_{x \to -1} \frac{1}{x^2 + 2x + 1} = \lim_{x \to -1} \frac{1}{(x + 1)^2} = \infty$$

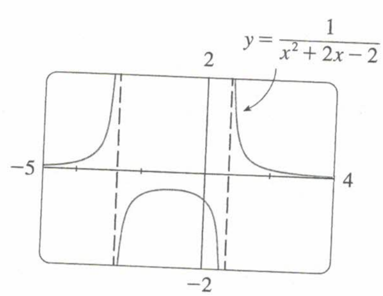


FIGURE 20
$c = -2$

When $c < 1$, there are two vertical asymptotes: $x = -1 \pm \sqrt{1 - c}$ (as in Figure

Now we compute the derivative:

$$f'(x) = -\frac{2x + 2}{(x^2 + 2x + c)^2}$$

This shows that $f'(x) = 0$ when $x = -1$ (if $c \neq 1$), $f'(x) > 0$ when $x < -1$, and $f'(x) < 0$ when $x > -1$. For $c \geqslant 1$, this means that f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$. For $c > 1$, there is an absolute maximum value $f(-1) = 1/(c - 1)$. For $c < 1$, $f(-1) = 1/(c - 1)$ is a local maximum value and intervals of increase and decrease are interrupted at the vertical asymptotes.

Figure 21 is a "slide show" displaying five members of the family, all graphed i viewing rectangle $[-5, 4]$ by $[-2, 2]$. As predicted, $c = 1$ is the value at which a t tion takes place from two vertical asymptotes to one, and then to none. As c increa from 1, we see that the maximum point becomes lower; this is explained by the fac $1/(c - 1) \to 0$ as $c \to \infty$. As c decreases from 1, the vertical asymptotes become n widely separated because the distance between them is $2\sqrt{1 - c}$, which becomes l

TEC See an animation of Figure 21 in Visual 4.4.

as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c-1) \to 0$ as $c \to -\infty$.

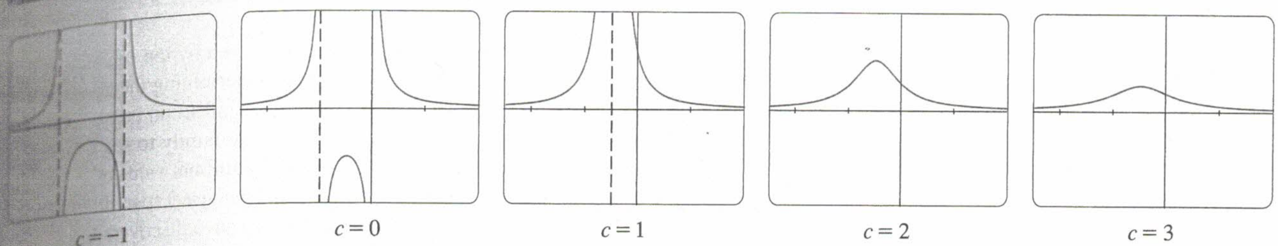


FIGURE 21 The family of functions $f(x) = 1/(x^2 + 2x + c)$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c-1)}/3$. So the inflection points become more spread out as c increases and this seems plausible from the last two parts of Figure 21. □

4.6 EXERCISES

1–8 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = 4x^4 - 32x^3 + 89x^2 - 95x + 29$

2. $f(x) = x^6 - 15x^5 + 75x^4 - 125x^3 - x$

3. $f(x) = x^6 - 10x^5 - 400x^4 + 2500x^3$

4. $f(x) = \dfrac{x^2 - 1}{40x^3 + x + 1}$

5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1}$

6. $f(x) = \tan x + 5 \cos x$

7. $f(x) = x^2 - 4x + 7 \cos x, \quad -4 \leqslant x \leqslant 4$

8. $f(x) = \dfrac{\sin x}{x}, \quad -2\pi \leqslant x \leqslant 2\pi$

9–10 Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease and intervals of concavity, and use calculus to find these intervals exactly.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$

10. $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

11–12 Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

11. $f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$

12. $f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$

CAS **13.** If f is the function considered in Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f'' and use it to estimate the intervals of concavity and inflection points.

CAS **14.** If f is the function of Exercise 12, find f' and f'' and use their graphs to estimate the intervals of increase and decrease and concavity of f.

CAS **15–18** Use a computer algebra system to graph f and to find f' and f''. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of f.

15. $f(x) = \dfrac{\sqrt{x}}{x^2 + x + 1}$

16. $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

17. $f(x) = \sqrt{x + 5 \sin x}, \quad x \leqslant 20$

18. $f(x) = \dfrac{2x - 1}{\sqrt[4]{x^4 + x + 1}}$

19. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing f in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graph of f'' at the same time. Find all the maximum and minimum values and inflection points. Then graph f in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

20–25 Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

20. $f(x) = x^3 + cx$

21. $f(x) = x^4 + cx^2$

22. $f(x) = x\sqrt{c^2 - x^2}$

23. $f(x) = \dfrac{cx}{1 + c^2x^2}$

24. $f(x) = \dfrac{1}{(1 - x^2)^2 + cx^2}$

25. $f(x) = cx + \sin x$

26. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

27. (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of c does the curve have minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

28. (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

4.7 OPTIMIZATION PROBLEMS

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section and the next we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 54 and adapt them to this situation:

STEPS IN SOLVING OPTIMIZATION PROBLEMS

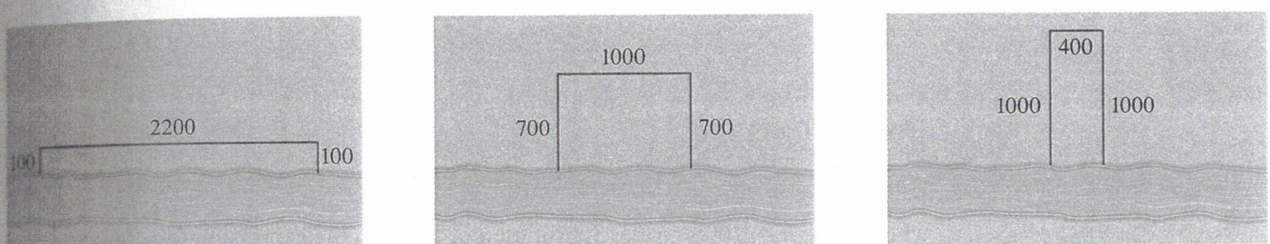
1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensior of the field that has the largest area?

- Understand the problem
- Analogy: Try special cases
- Draw diagrams

SOLUTION In order to get a feeling for what is happening in this problem, let's experimer with some special cases. Figure 1 (not to scale) shows three possible ways of laying o the 2400 ft of fencing.



Area = 100 · 2200 = 220,000 ft²   Area = 700 · 1000 = 700,000 ft²   Area = 1000 · 400 = 400,000 ft²

FIGURE 1

We see that when we try shallow, wide fields or deep, narrow fields, we get relative small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

- Introduce notation

Figure 2 illustrates the general case. We wish to maximize the area A of the rectang Let x and y be the depth and width of the rectangle (in feet). Then we express A in ter of x and y:

$$A = xy$$

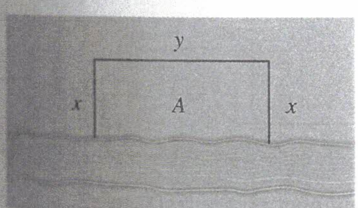


FIGURE 2

We want to express A as a function of just one variable, so we eliminate y by expressi it in terms of x. To do this we use the given information that the total length of the fen ing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Note that $x \geqslant 0$ and $x \leqslant 1200$ (otherwise $A < 0$). So the function that we wish to ma mize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leqslant x \leqslant 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

Thus the rectangular field should be 600 ft deep and 1200 ft wide.

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

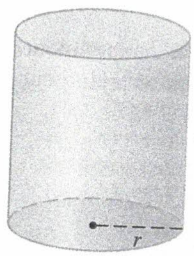


FIGURE 3

SOLUTION Draw the diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h we use the fact that the volume is given as 1 L, which we take to be 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

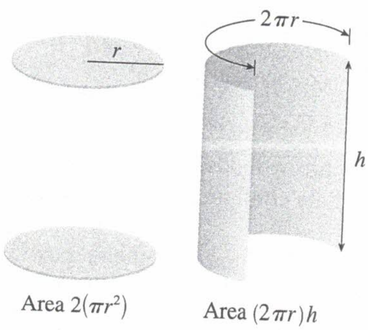


FIGURE 4

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

■ In the Applied Project on page 268 we investigate the most economical shape for a can by taking into account other manufacturing costs.

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for *all* r to the left of the critical number and increasing for *all* r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

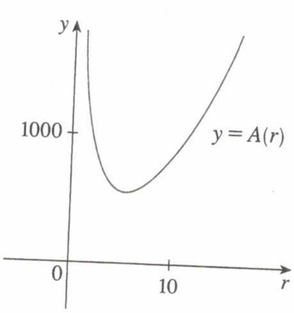


FIGURE 5

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. □

NOTE 1 The argument used in Example 2 to justify the absolute minimum is a variant of the First Derivative Test (which applies only to *local* maximum or minimum values) and is stated here for future reference.

> **FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREME VALUES** Suppose that c is a critical number of a continuous function f defined on an interval.
>
> (a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.
>
> (b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

TEC Module 4.7 takes you through six additional optimization problems, including animations of the physical situations.

NOTE 2 An alternative method for solving optimization problems is to use implicit differentiation. Let's look at Example 2 again to illustrate the method. We work with the same equations

$$A = 2\pi r^2 + 2\pi rh \qquad \pi r^2 h = 1000$$

but instead of eliminating h, we differentiate both equations implicitly with respect to r:

$$A' = 4\pi r + 2\pi h + 2\pi rh' \qquad 2\pi rh + \pi r^2 h' = 0$$

The minimum occurs at a critical number, so we set $A' = 0$, simplify, and arrive at the equations

$$2r + h + rh' = 0 \qquad 2h + rh' = 0$$

and subtraction gives $2r - h = 0$, or $h = 2r$.

V EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$

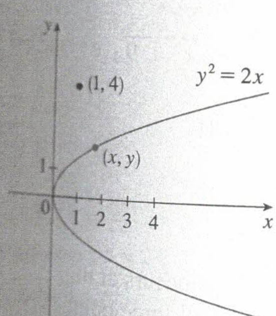


FIGURE 6

(See Figure 6.) But if (x, y) lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.) Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y - 4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$.

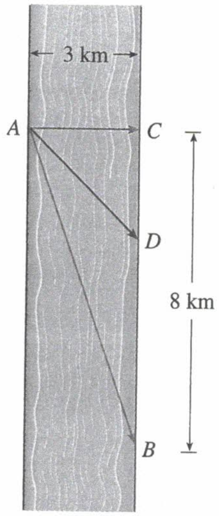


FIGURE 7

EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

SOLUTION If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9}/6$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x = 0$, he rows to C and if $x = 8$, he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geqslant 0$, we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9}$$

$$\iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81$$

$$\iff x = \frac{9}{\sqrt{7}}$$

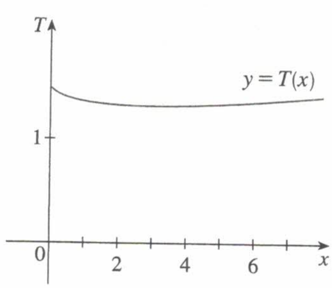


FIGURE 8

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we evaluate T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Figure 8 illustrates this calculation by showing the graph of T.

Thus the man should land the boat at a point $9/\sqrt{7}$ km ($\approx$3.4 km) downstream from his starting point. □

V EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

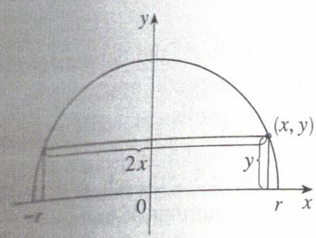


FIGURE 9

SOLUTION 1 Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the x-axis as shown in Figure 9.

Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$$A = 2xy$$

To eliminate y we use the fact that (x, y) lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of x gives a maximum value of A since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

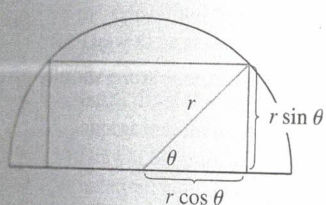


FIGURE 10

SOLUTION 2 A simpler solution is possible if we think of using an angle as a variable. Let θ be the angle shown in Figure 10. Then the area of the rectangle is

$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\,\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of r^2 and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all. □

APPLICATIONS TO BUSINESS AND ECONOMICS

In Section 3.7 we introduced the idea of marginal cost. Recall that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x. In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x. If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function. In Exercises 53–58 you are asked to use the marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

EXAMPLE 6 A store has been selling 200 DVD burners a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

SOLUTION If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of \$125. □

4.7 EXERCISES

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

8. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

9. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the volume as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

11. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

12. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

13. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

15. Do Exercise 14 assuming the container has a lid that is made from the same material as the sides.

16. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

17. Find the point on the line $y = 4x + 7$ that is closest to the origin.

18. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

19. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

20. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

21. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

22. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

23. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

24. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

25. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

26. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

27. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

28. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

29. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

30. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 56 on page 23.) If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

31. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

32. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?

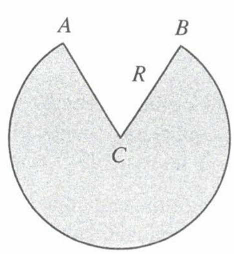
33. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

34. Answer Exercise 33 if one piece is bent into a square and the other into a circle.

35. A cylindrical can without a top is made to contain $V\ \text{cm}^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

36. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

37. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



38. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

39. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

40. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

41. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

42. For a fish swimming at a speed v relative to the water, t energy expenditure per unit time is proportional to v^3. I believed that migrating fish try to minimize the total en required to swim a fixed distance. If the fish are swimm against a current u ($u < v$), then the time required to sw distance L is $L/(v - u)$ and the total energy E required swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

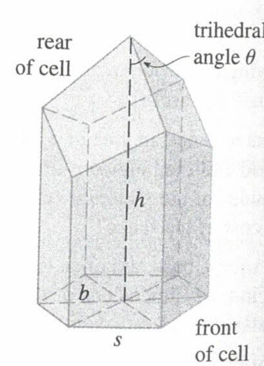
Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% gre than the current speed.

43. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the ure. It is believed that bees form their cells in such a way minimize the surface area for a given volume, thus using least amount of wax in cell construction. Examination of cells has shown that the measure of the apex angle θ is ar ingly consistent. Based on the geometry of the cell, it can shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot \theta + \left(3s^2\sqrt{3}/2\right) \csc \theta$$

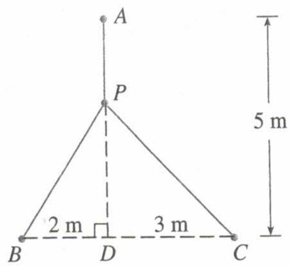
where s, the length of the sides of the hexagon, and h, the height, are constants.
(a) Calculate $dS/d\theta$.
(b) What angle should the bees prefer?
(c) Determine the minimum surface area of the cell (in ter of s and h).

Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.

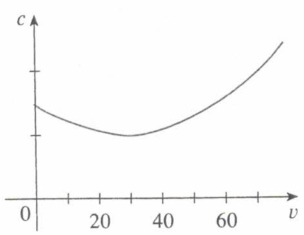


44. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east a 15 km/h and reaches the same dock at 3:00 PM. At what tim were the two boats closest together?

and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value.



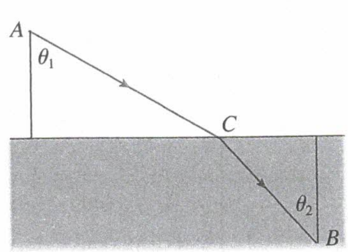
62. The graph shows the fuel consumption c of a car (measured in gallons per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 30$ mi/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in gallons *per mile*. Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



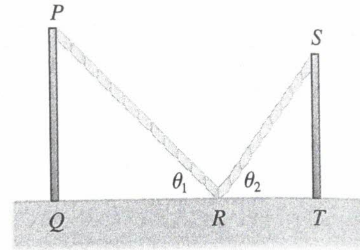
63. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

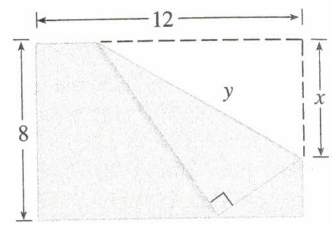
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



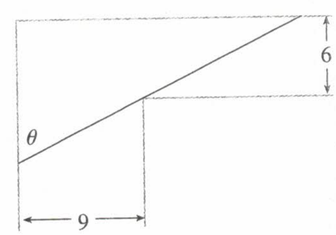
64. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



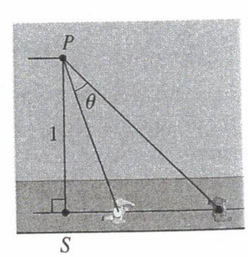
65. The upper right-hand corner of a piece of paper, 12 in. by 8 in., as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



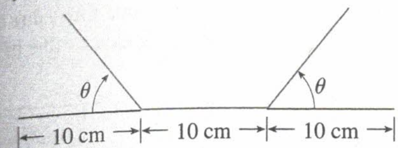
66. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



67. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners. [*Hint:* Maximize $\tan\theta$.]



68. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?

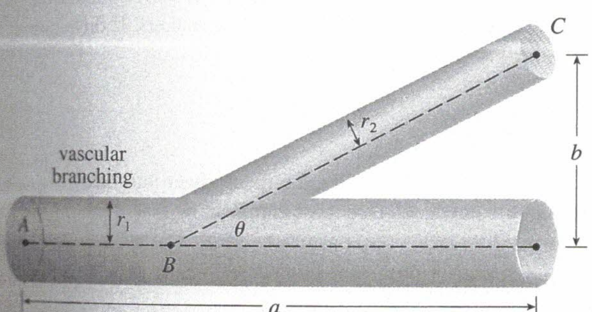


69. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W. [*Hint:* Express the area as a function of an angle θ.]

70. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 9.4.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

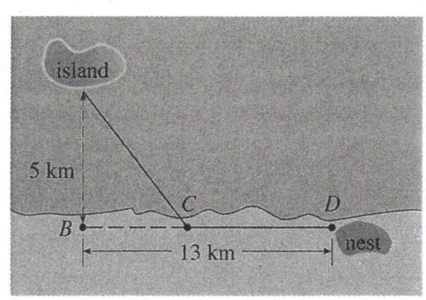
71. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?



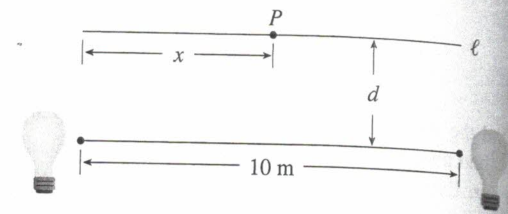
72. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

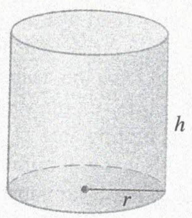
(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.



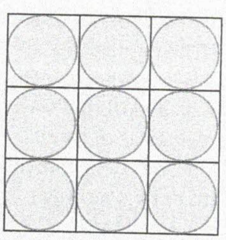
APPLIED PROJECT — THE SHAPE OF A CAN



In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 2 in Section 4.7 and we found that $h = 2r$; that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this phenomenon.

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side $2r$ (as in the figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

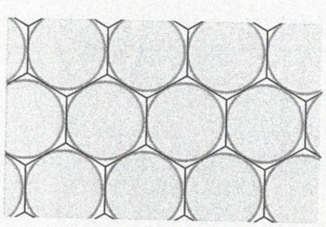
$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$



Discs cut from squares

2. A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$



Discs cut from hexagons

3. The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate the manufacturing of the can into the cost. Let's assume that most of the expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}\,r^2 + 2\pi rh + k(4\pi r + h)$$

where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$$

4. Plot $\sqrt[3]{V}/k$ as a function of $x = h/r$ and use your graph to argue that when a can is large or joining is cheap, we should make h/r approximately 2.21 (as in Problem 2). But when the can is small or joining is costly, h/r should be substantially larger.

5. Our analysis shows that large cans should be almost square but small cans should be tall and thin. Take a look at the relative shapes of the cans in a supermarket. Is our conclusion usually true in practice? Are there exceptions? Can you suggest reasons why small cans are not always tall and thin?

4.8 NEWTON'S METHOD

Suppose that a car dealer offers to sell you a car for \$18,000 or for payments of \$375 per month for five years. You would like to know what monthly interest rate the dealer is, in effect, charging you. To find the answer, you have to solve the equation

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0 \tag{1}$$

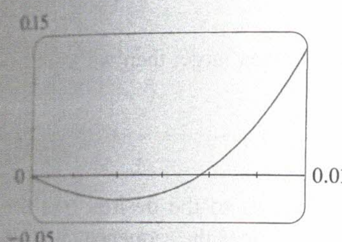


FIGURE 1

(The details are explained in Exercise 39.) How would you solve such an equation?

For a quadratic equation $ax^2 + bx + c = 0$ there is a well-known formula for the roots. For third- and fourth-degree equations there are also formulas for the roots, but they are extremely complicated. If f is a polynomial of degree 5 or higher, there is no such formula (see the note on page 167). Likewise, there is no formula that will enable us to find the exact roots of a transcendental equation such as $\cos x = x$.

We can find an *approximate* solution to Equation 1 by plotting the left side of the equation. Using a graphing device, and after experimenting with viewing rectangles, we produce the graph in Figure 1.

We see that in addition to the solution $x = 0$, which doesn't interest us, there is a solution between 0.007 and 0.008. Zooming in shows that the root is approximately 0.0076. If we need more accuracy we could zoom in repeatedly, but that becomes tiresome. A faster alternative is to use a numerical rootfinder on a calculator or computer algebra system. If we do so, we find that the root, correct to nine decimal places, is 0.007628603.

■ Try to solve Equation 1 using the numerical rootfinder on your calculator or computer. Some machines are not able to solve it. Others are successful but require you to specify a starting point for the search.

How do those numerical rootfinders work? They use a variety of methods, but most of them make some use of **Newton's method**, also called the **Newton-Raphson method**. We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled r. We start with a first approximation x_1, which is obtained by guessing, or from a rough sketch of the graph of f, or from a computer-generated graph of f. Consider the tangent line L to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the x-intercept of L, labeled x_2. The idea behind Newton's method is that the tangent line is close to the curve and so its x-intercept, x_2, is close to the x-intercept of the curve (namely, the root r that we are seeking). Because the tangent is a line, we can easily find its x-intercept.

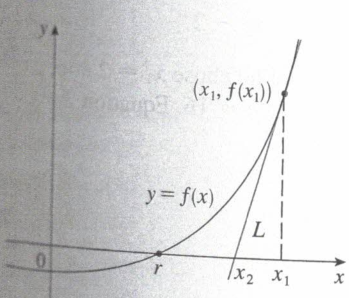


FIGURE 2

To find a formula for x_2 in terms of x_1 we use the fact that the slope of L is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept of L is x_2, we set $y = 0$ and obtain

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use x_2 as a second approximation to r.

Next we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

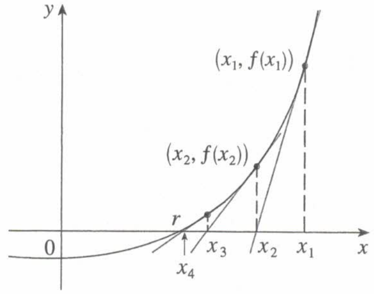


FIGURE 3

If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in Figure 3. In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

2
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

■ Sequences were briefly introduced in *A Preview of Calculus* on page 6. A more thorough discussion starts in Section 12.1.

Although the sequence of successive approximations converges to the desired root for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge. For example, consider the situation shown in Figure 4. You can see that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation x_1 should be chosen. See Exercises 29–32 for specific examples in which Newton's method works very slowly or does not work at all.

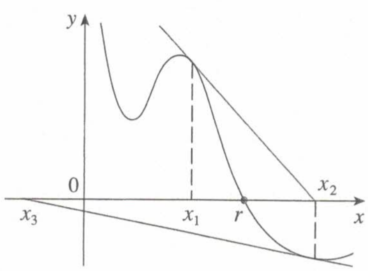


FIGURE 4

EXAMPLE 1 Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

SOLUTION We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Equation 2 becomes

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

TEC In Module 4.8 you can investigate how Newton's Method works for several functions and what happens when you change x_1.

With $n = 1$ we have

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

Figure 5 shows the geometry behind the first step in Newton's method in Example 1. Since $f'(2) = 10$, the tangent line to $y = x^3 - 2x - 5$ at $(2, -1)$ has equation $y = 10x - 21$ so its x-intercept is $x_2 = 2.1$.

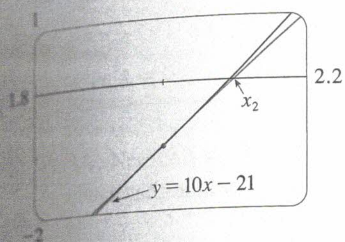


FIGURE 5

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places. □

Suppose that we want to achieve a given accuracy, say to eight decimal places, using Newton's method. How do we know when to stop? The rule of thumb that is generally used is that we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places. (A precise statement concerning accuracy in Newton's method will be given in Exercise 39 in Section 12.11.)

Notice that the procedure in going from n to $n + 1$ is the same for all values of n. (It is called an *iterative* process.) This means that Newton's method is particularly convenient for use with a programmable calculator or a computer.

EXAMPLE 2 Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

SOLUTION First we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation

$$x^6 - 2 = 0$$

so we take $f(x) = x^6 - 2$. Then $f'(x) = 6x^5$ and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$x_2 \approx 1.16666667$$
$$x_3 \approx 1.12644368$$
$$x_4 \approx 1.12249707$$
$$x_5 \approx 1.12246205$$
$$x_6 \approx 1.12246205$$

Since x_5 and x_6 agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places. □

EXAMPLE 3 Find, correct to six decimal places, the root of the equation $\cos x = x$.

SOLUTION We first rewrite the equation in standard form:

$$\cos x - x = 0$$

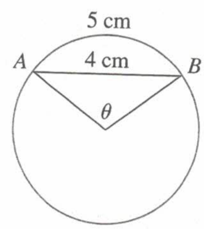
Therefore we let $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$, so Formula 2 becomes

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

36. Of the infinitely many lines that are tangent to the curve $y = -\sin x$ and pass through the origin, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.

37. Use Newton's method to find the coordinates, correct to six decimal places, of the point on the parabola $y = (x - 1)^2$ that is closest to the origin.

38. In the figure, the length of the chord AB is 4 cm and the length of the arc AB is 5 cm. Find the central angle θ, in radians, correct to four decimal places. Then give the answer to the nearest degree.



39. A car dealer sells a new car for \$18,000. He also offers to sell the same car for payments of \$375 per month for five years. What monthly interest rate is this dealer charging?

To solve this problem you will need to use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

$$A = \frac{R}{i}\left[1 - (1 + i)^{-n}\right]$$

Replacing i by x, show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

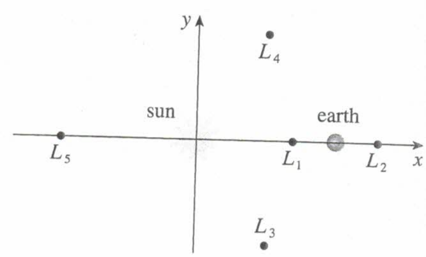
40. The figure shows the sun located at the origin and the earth at the point $(1, 0)$. (The unit here is the distance between the centers of the earth and the sun, called an *astronomical unit*: $1 \text{ AU} \approx 1.496 \times 10^8$ km.) There are five locations $L_1, L_2, L_3,$ L_4, and L_5 in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite (including the gravitational attractions of the earth and the sun) balance each other. These locations are called *libration points*. (A solar research satellite has been placed at one of these libration points.) If m_1 is the mass of the sun, m_2 is the mass of the earth, and $r = m_2/(m_1 + m_2)$, it turns out that the x-coordinate of L_1 is the unique root of the fifth-degree equation

$$p(x) = x^5 - (2 + r)x^4 + (1 + 2r)x^3 - (1 - r)x^2 + 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the root of the equation

$$p(x) - 2rx^2 = 0$$

Using the value $r \approx 3.04042 \times 10^{-6}$, find the locations of the libration points (a) L_1 and (b) L_2.



4.9 ANTIDERIVATIVES

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

> **DEFINITION** A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. The question arises: Are there any others?

To answer this question, recall that in Section 4.2 we used the Mean Value Theorem to prove that if two functions have identical derivatives on an interval, then they must differ by a constant (Corollary 4.2.7). Thus if F and G are any two antiderivatives of f, then

$$F'(x) = f(x) = G'(x)$$

so $G(x) - F(x) = C$, where C is a constant. We can write this as $G(x) = F(x) + C$, so we have the following result.

> **1 THEOREM** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

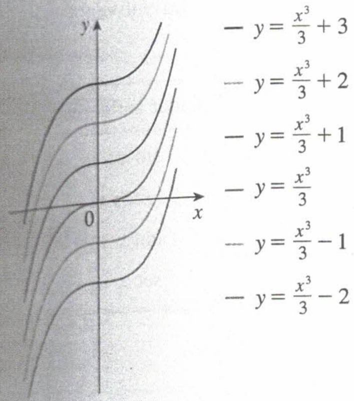


FIGURE 1
Members of the family of antiderivatives of $f(x) = x^2$

Going back to the function $f(x) = x^2$, we see that the general antiderivative of f is $\frac{1}{3}x^3 + C$. By assigning specific values to the constant C, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1). This makes sense because each curve must have the same slope at any given value of x.

EXAMPLE 1 Find the most general antiderivative of each of the following functions.
(a) $f(x) = \sin x$ (b) $f(x) = x^n, \quad n \geqslant 0$ (c) $f(x) = x^{-3}$

SOLUTION
(a) If $F(x) = -\cos x$, then $F'(x) = \sin x$, so an antiderivative of $\sin x$ is $-\cos x$. By Theorem 1, the most general antiderivative is $G(x) = -\cos x + C$.

(b) We use the Power Rule to discover an antiderivative of x^n:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n$$

Thus the general antiderivative of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

This is valid for $n \geqslant 0$ because then $f(x) = x^n$ is defined on an interval.

(c) If we put $n = -3$ in part (b) we get the particular antiderivative $F(x) = x^{-2}/(-2)$ by the same calculation. But notice that $f(x) = x^{-3}$ is not defined at $x = 0$. Thus Theorem 1 tells us only that the general antiderivative of f is $x^{-2}/(-2) + C$ on any interval that does not contain 0. So the general antiderivative of $f(x) = 1/x^3$ is

$$F(x) = \begin{cases} -\dfrac{1}{2x^2} + C_1 & \text{if } x > 0 \\ -\dfrac{1}{2x^2} + C_2 & \text{if } x < 0 \end{cases}$$ □

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

2 TABLE OF ANTIDIFFERENTIATION FORMULAS

■ To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

EXAMPLE 2 Find all functions g such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

SOLUTION We first rewrite the given function as follows:

$$g'(x) = 4 \sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4 \sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4 \sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$g(x) = 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4 \cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$

□

In applications of calculus it is very common to have a situation as in Example 2, where it is required to find a function, given knowledge about its derivatives. An equation that involves the derivatives of a function is called a **differential equation**. These will be studied in some detail in Chapter 10, but for the present we can solve some elementary differential equations. The general solution of a differential equation involves an arbitrary constant (or constants) as in Example 2. However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

EXAMPLE 3 Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

SOLUTION The general antiderivative of

$$f'(x) = x^{3/2}$$

is

$$f(x) = \frac{x^{5/2}}{\frac{5}{2}} + C = \tfrac{2}{5}x^{5/2} + C$$

To determine C we use the fact that $f(1) = 2$:

$$f(1) = \tfrac{2}{5} + C = 2$$

Solving for C, we get $C = 2 - \frac{2}{5} = \frac{8}{5}$, so the particular solution is

$$f(x) = \frac{2x^{5/2} + 8}{5}$$ □

EXAMPLE 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

SOLUTION The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$ □

If we are given the graph of a function f, it seems reasonable that we should be able to sketch the graph of an antiderivative F. Suppose, for instance, that we are given that $F(0) = 1$. Then we have a place to start, the point $(0, 1)$, and the direction in which we move our pencil is given at each stage by the derivative $F'(x) = f(x)$. In the next example we use the principles of this chapter to show how to graph F even when we don't have a formula for f. This would be the case, for instance, when $f(x)$ is determined by experimental data.

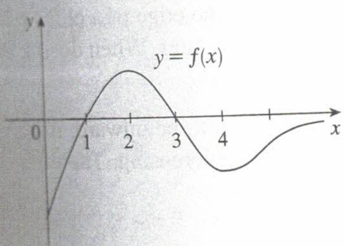


FIGURE 2

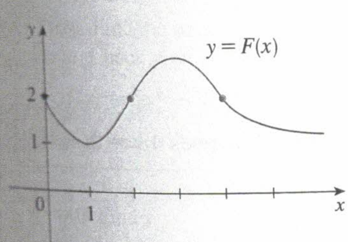


FIGURE 3

EXAMPLE 5 The graph of a function f is given in Figure 2. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.

SOLUTION We are guided by the fact that the slope of $y = F(x)$ is $f(x)$. We start at the point $(0, 2)$ and draw F as an initially decreasing function since $f(x)$ is negative when $0 < x < 1$. Notice that $f(1) = f(3) = 0$, so F has horizontal tangents when $x = 1$ and $x = 3$. For $1 < x < 3$, $f(x)$ is positive and so F is increasing. We see that F has a local minimum when $x = 1$ and a local maximum when $x = 3$. For $x > 3$, $f(x)$ is negative and so F is decreasing on $(3, \infty)$. Since $f(x) \to 0$ as $x \to \infty$, the graph of F becomes flatter as $x \to \infty$. Also notice that $F''(x) = f'(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$, so F has inflection points when $x = 2$ and $x = 4$. We use this information to sketch the graph of the antiderivative in Figure 3. □

RECTILINEAR MOTION

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$. This means that the position function is an antiderivative of the velocity function. Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration. If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

V EXAMPLE 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$ □

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by g. For motion close to the ground we may assume that g is constant, its value being about 9.8 m/s² (or 32 ft/s²).

EXAMPLE 7 A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

SOLUTION The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine C we use the given information that $v(0) = 48$. This gives $48 = 0 + C$, so

$$v(t) = -32t + 48$$

The maximum height is reached when $v(t) = 0$, that is, after 1.5 s. Since $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -16t^2 + 48t + D$$

Using the fact that $s(0) = 432$, we have $432 = 0 + D$ and so

$$s(t) = -16t^2 + 48t + 432$$

Figure 4 shows the position function of the ball in Example 7. The graph corroborates the conclusions we reached: The ball reaches its maximum height after 1.5 s and hits the ground after 6.9 s.

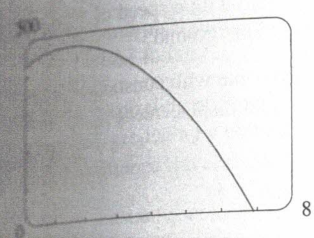


FIGURE 4

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-16t^2 + 48t + 432 = 0$$

or, equivalently,

$$t^2 - 3t - 27 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{3 \pm 3\sqrt{13}}{2}$$

We reject the solution with the minus sign since it gives a negative value for t. Therefore the ball hits the ground after $3(1 + \sqrt{13})/2 \approx 6.9$ s. □

4.9 EXERCISES

1–18 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = x - 3$

2. $f(x) = \frac{1}{2}x^2 - 2x + 6$

3. $f(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$

4. $f(x) = 8x^9 - 3x^6 + 12x^3$

5. $f(x) = (x + 1)(2x - 1)$

6. $f(x) = x(2 - x)^2$

7. $f(x) = 5x^{1/4} - 7x^{3/4}$

8. $f(x) = 2x + 3x^{1.7}$

9. $f(x) = 6\sqrt{x} - \sqrt[6]{x}$

10. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$

11. $f(x) = \dfrac{10}{x^9}$

12. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$

13. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2}$

14. $f(t) = 3\cos t - 4\sin t$

15. $g(\theta) = \cos\theta - 5\sin\theta$

16. $f(\theta) = 6\theta^2 - 7\sec^2\theta$

17. $f(t) = 2\sec t\tan t + \frac{1}{2}t^{-1/2}$

18. $f(x) = 2\sqrt{x} + 6\cos x$

19–20 Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

19. $f(x) = 5x^4 - 2x^5$, $F(0) = 4$

20. $f(x) = x + 2\sin x$, $F(0) = -6$

21–40 Find f.

21. $f''(x) = 6x + 12x^2$

22. $f''(x) = 2 + x^3 + x^6$

23. $f''(x) = \frac{2}{3}x^{2/3}$

24. $f''(x) = 6x + \sin x$

25. $f'''(t) = 60t^2$

26. $f'''(t) = t - \sqrt{t}$

27. $f'(x) = 1 - 6x$, $f(0) = 8$

28. $f'(x) = 8x^3 + 12x + 3$, $f(1) = 6$

29. $f'(x) = \sqrt{x}(6 + 5x)$, $f(1) = 10$

30. $f'(x) = 2x - 3/x^4$, $x > 0$, $f(1) = 3$

31. $f'(t) = 2\cos t + \sec^2 t$, $-\pi/2 < t < \pi/2$, $f(\pi/3) = 4$

32. $f'(x) = x^{-1/3}$, $f(1) = 1$, $f(-1) = -1$

33. $f''(x) = 24x^2 + 2x + 10$, $f(1) = 5$, $f'(1) = -3$

34. $f''(x) = 4 - 6x - 40x^3$, $f(0) = 2$, $f'(0) = 1$

35. $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3$, $f'(0) = 4$

36. $f''(t) = 3/\sqrt{t}$, $f(4) = 20$, $f'(4) = 7$

37. $f''(x) = 2 - 12x$, $f(0) = 9$, $f(2) = 15$

38. $f''(x) = 20x^3 + 12x^2 + 4$, $f(0) = 8$, $f(1) = 5$

39. $f''(x) = 2 + \cos x$, $f(0) = -1$, $f(\pi/2) = 0$

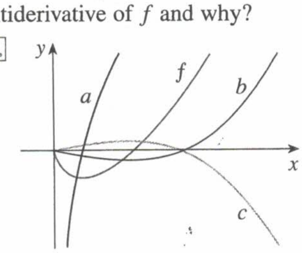
40. $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$

41. Given that the graph of f passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(2)$.

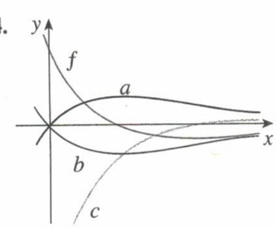
42. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

43–44 The graph of a function f is shown. Which graph is an antiderivative of f and why?

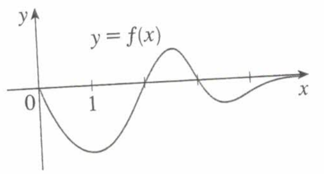
43.



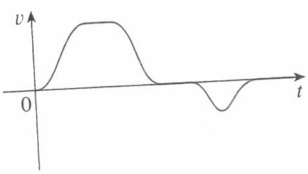
44.



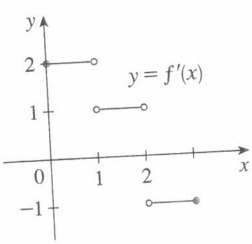
45. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.



46. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of the position function.



47. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous and $f(0) = -1$.



48. (a) Use a graphing device to graph $f(x) = 2x - 3\sqrt{x}$.
(b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies $F(0) = 1$.
(c) Use the rules of this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare with your sketch in part (b).

49–50 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

49. $f(x) = \dfrac{\sin x}{1 + x^2}, \quad -2\pi \leqslant x \leqslant 2\pi$

50. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 1, \quad -1.5 \leqslant x \leqslant 1.5$

51–56 A particle is moving with the given data. Find the position of the particle.

51. $v(t) = \sin t - \cos t, \quad s(0) = 0$

52. $v(t) = 1.5\sqrt{t}, \quad s(4) = 10$

53. $a(t) = t - 2, \quad s(0) = 1, \quad v(0) = 3$

54. $a(t) = \cos t + \sin t, \quad s(0) = 0, \quad v(0) = 5$

55. $a(t) = 10 \sin t + 3 \cos t, \quad s(0) = 0, \quad s(2\pi) = 12$

56. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

57. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

58. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0$$

59. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

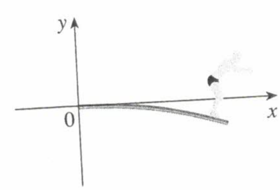
60. Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 48 ft/s and the other is thrown a second later with a speed of 24 ft/s. Do the balls ever pass each other?

61. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

62. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of a curve $y = f(x)$, where

$$EIy'' = mg(L - x) + \tfrac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and g (< 0) is the acceleration due to gravity.
(a) Find an expression for the shape of the curve.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.



63. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

64. The linear density of a rod of length 1 m is given by $\rho(x) = 1/\sqrt{x}$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.

65. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \leq t \leq 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

66. A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

67. What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 s?

68. A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

69. A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a pileup?

70. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 60t$, at which time the fuel is exhausted and it becomes a freely "falling" body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -18 ft/s in 5 s. The rocket then "floats" to the ground at that rate.
   (a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
   (b) At what time does the rocket reach its maximum height, and what is that height?
   (c) At what time does the rocket land?

71. A high-speed bullet train accelerates and decelerates at the rate of 4 ft/s^2. Its maximum cruising speed is 90 mi/h.
   (a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
   (b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
   (c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 miles apart.
   (d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

4 REVIEW

CONCEPT CHECK

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.
2. (a) What does the Extreme Value Theorem say?
   (b) Explain how the Closed Interval Method works.
3. (a) State Fermat's Theorem.
   (b) Define a critical number of f.
4. (a) State Rolle's Theorem.
   (b) State the Mean Value Theorem and give a geometric interpretation.
5. (a) State the Increasing/Decreasing Test.
   (b) What does it mean to say that f is concave upward on an interval I?
   (c) State the Concavity Test.
   (d) What are inflection points? How do you find them?
6. (a) State the First Derivative Test.
   (b) State the Second Derivative Test.
   (c) What are the relative advantages and disadvantages of these tests?
7. Explain the meaning of each of the following statements.
   (a) $\lim_{x \to \infty} f(x) = L$ (b) $\lim_{x \to -\infty} f(x) = L$ (c) $\lim_{x \to \infty} f(x) = \infty$
   (d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.
8. If you have a graphing calculator or computer, why do you need calculus to graph a function?
9. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.
   (b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.
   (c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
   (d) Under what circumstances is Newton's method likely to fail or to work very slowly?
10. (a) What is an antiderivative of a function f?
    (b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.
2. If f has an absolute minimum value at c, then $f'(c) = 0$.
3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).
4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.
5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.
6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.
7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.
8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.
9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.
12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.
13. If f and g are increasing on an interval I, then fg is inc on I.
14. If f and g are positive increasing functions on an interv then fg is increasing on I.
15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x$ decreasing on I.
16. If f is even, then f' is even.
17. If f is periodic, then f' is periodic.
18. The most general antiderivative of $f(x) = x^{-2}$ is

$$F(x) = -\frac{1}{x} + C$$

19. If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.

EXERCISES

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 6x^2 + 9x + 1, \quad [2, 4]$
2. $f(x) = x\sqrt{1 - x}, \quad [-1, 1]$
3. $f(x) = \dfrac{3x - 4}{x^2 + 1}, \quad [-2, 2]$
4. $f(x) = (x^2 + 2x)^3, \quad [-2, 1]$
5. $f(x) = x + \sin 2x, \quad [0, \pi]$
6. $f(x) = \sin x + \cos^2 x, \quad [0, \pi]$

7–12 Find the limit.

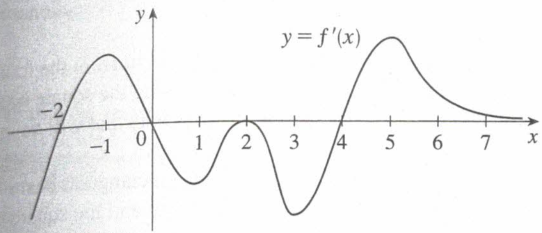
7. $\displaystyle\lim_{x \to \infty} \frac{3x^4 + x - 5}{6x^4 - 2x^2 + 1}$
8. $\displaystyle\lim_{t \to \infty} \frac{t^3 - t + 2}{(2t - 1)(t^2 + t + 1)}$
9. $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1}$
10. $\displaystyle\lim_{x \to -\infty} (x^2 + x^3)$
11. $\displaystyle\lim_{x \to \infty} \left(\sqrt{4x^2 + 3x} - 2x\right)$
12. $\displaystyle\lim_{x \to \infty} \frac{\sin^4 x}{\sqrt{x}}$

13–15 Sketch the graph of a function that satisfies the given conditions:

13. $f(0) = 0, \quad f'(-2) = f'(1) = f'(9) = 0,$
$\lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to 6} f(x) = -\infty,$
$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,
$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,
$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,
$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$
14. $f(0) = 0, \quad f$ is continuous and even,
$f'(x) = 2x$ if $0 < x < 1, \quad f'(x) = -1$ if $1 < x < 3,$
$f'(x) = 1$ if $x > 3$
15. f is odd, $\quad f'(x) < 0$ for $0 < x < 2,$
$f'(x) > 0$ for $x > 2, \quad f''(x) > 0$ for $0 < x < 3,$
$f''(x) < 0$ for $x > 3, \quad \lim_{x\to\infty} f(x) = -2$

16. The figure shows the graph of the *derivative* f' of a functi
(a) On what intervals is f increasing or decreasing?
(b) For what values of x does f have a local maximum or minimum?

(c) Sketch the graph of f''.
(d) Sketch a possible graph of f.



17–28 Use the guidelines of Section 4.5 to sketch the curve.

17. $y = 2 - 2x - x^3$

18. $y = x^3 - 6x^2 - 15x + 4$

19. $y = x^4 - 3x^3 + 3x^2 - x$

20. $y = \dfrac{1}{1 - x^2}$

21. $y = \dfrac{1}{x(x - 3)^2}$

22. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

23. $y = x^2/(x + 8)$

24. $y = \sqrt{1 - x} + \sqrt{1 + x}$

25. $y = x\sqrt{2 + x}$

26. $y = \sqrt[3]{x^2 + 1}$

27. $y = \sin^2 x - 2\cos x$

28. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$

29–32 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 29 use calculus to find these quantities exactly.

29. $f(x) = \dfrac{x^2 - 1}{x^3}$

30. $f(x) = \dfrac{x^3 - x}{x^2 + x + 3}$

31. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

32. $f(x) = x^2 + 6.5 \sin x, \quad -5 \leq x \leq 5$

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33. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root.

34. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$. Show that $9 \leq f(4) \leq 21$.

35. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval $[32, 33]$, show that

$$2 < \sqrt[5]{33} < 2.0125$$

36. For what values of the constants a and b is $(1, 6)$ a point of inflection of the curve $y = x^3 + ax^2 + bx + 1$?

37. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
(a) At what numbers does g have an extreme value?
(b) Discuss the concavity of g.

38. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

39. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

40. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

41. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

42. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

43. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

44. Solve Exercise 43 when $|CD| = 2$ cm.

45. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

46. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

47. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

48. A manufacturer determines that the cost of making x units of a commodity is $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and the demand function is $p(x) = 48.2 - 0.03x$.
(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
(b) Use calculus to find the production level for maximum profit.
(c) Estimate the production level that minimizes the average cost.

49. Use Newton's method to find the root of the equation $x^5 - x^4 + 3x^2 - 3x - 2 = 0$ in the interval $[1, 2]$ correct to six decimal places.

50. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

51. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

52. Use the guidelines in Section 4.5 to sketch the curve $y = x \sin x$, $0 \leqslant x \leqslant 2\pi$. Use Newton's method when necessary.

53–58 Find f.

53. $f'(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

54. $f'(x) = 8x - 3 \sec^2 x$

55. $f'(t) = 2t - 3 \sin t, \quad f(0) = 5$

56. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}, \quad f(1) = 3$

57. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

58. $f''(x) = 2x^3 + 3x^2 - 4x + 5, \quad f(0) = 2, \quad f(1) = 0$

59–60 A particle is moving with the given data. Find the position of the particle.

59. $v(t) = 2t - \sin t, \quad s(0) = 3$

60. $a(t) = \sin t + 3 \cos t, \quad s(0) = 0, \quad v(0) = 2$

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61. Use a graphing device to draw a graph of the function $f(x) = x^2 \sin(x^2)$, $0 \leqslant x \leqslant \pi$, and use that graph to sketch the antiderivative F of f that satisfies the initial condition $F(0) = 0$.

62. Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

63. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

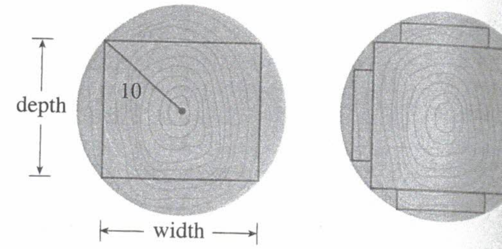
64. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

65. A rectangular beam will be cut from a cylindrical log radius 10 inches.

(a) Show that the beam of maximal cross-sectional ar a square.

(b) Four rectangular planks will be cut from the four s of the log that remain after cutting the square beam mine the dimensions of the planks that will have m cross-sectional area.

(c) Suppose that the strength of a rectangular beam is tional to the product of its width and the square of depth. Find the dimensions of the strongest beam t be cut from the cylindrical log.



66. If a projectile is fired with an initial velocity v at an ang inclination θ from the horizontal, then its trajectory, neg ing air resistance, is the parabola

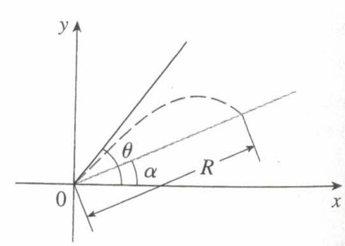
$$y = (\tan \theta)x - \frac{g}{2v^2 \cos^2\theta} x^2 \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a pla that is inclined at an angle α, $\alpha > 0$, from the horizo as shown in the figure. Show that the range of the pr tile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2 \cos\theta \, \sin(\theta - \alpha)}{g \cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

(c) Suppose the plane is at an angle α *below* the horizont Determine the range R in this case, and determine the angle at which the projectile should be fired to maximi



One of the most important principles of problem solving is *analogy* (see page 54). If you are having trouble getting started on a problem, it is sometimes helpful to start by solving a similar, but simpler, problem. The following example illustrates the principle. Cover up the solution and try solving it yourself first.

EXAMPLE 1 If x, y, and z are positive numbers, prove that

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geqslant 8$$

SOLUTION It may be difficult to get started on this problem. (Some students have tackled it by multiplying out the numerator, but that just creates a mess.) Let's try to think of a similar, simpler problem. When several variables are involved, it's often helpful to think of an analogous problem with fewer variables. In the present case we can reduce the number of variables from three to one and prove the analogous inequality

$$\boxed{1} \qquad \frac{x^2+1}{x} \geqslant 2 \qquad \text{for } x > 0$$

In fact, if we are able to prove (1), then the desired inequality follows because

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} = \left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)\left(\frac{z^2+1}{z}\right) \geqslant 2 \cdot 2 \cdot 2 = 8$$

The key to proving (1) is to recognize that it is a disguised version of a minimum problem. If we let

$$f(x) = \frac{x^2+1}{x} = x + \frac{1}{x} \qquad x > 0$$

then $f'(x) = 1 - (1/x^2)$, so $f'(x) = 0$ when $x = 1$. Also, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. Therefore the absolute minimum value of f is $f(1) = 2$. This means that

$$\frac{x^2+1}{x} \geqslant 2 \qquad \text{for all positive values of } x$$

and, as previously mentioned, the given inequality follows by multiplication.

The inequality in (1) could also be proved without calculus. In fact, if $x > 0$, we have

$$\frac{x^2+1}{x} \geqslant 2 \iff x^2+1 \geqslant 2x \iff x^2 - 2x + 1 \geqslant 0$$

$$\iff (x-1)^2 \geqslant 0$$

Because the last inequality is obviously true, the first one is true too. □

Look Back

What have we learned from the solution to this example?

- To solve a problem involving several variables, it might help to solve a similar problem with just one variable.
- When trying to prove an inequality, it might help to think of it as a maximum or minimum problem.

PROBLEMS PLUS

PROBLEMS

1. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

2. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant$

3. Let a and b be positive numbers. Show that not both of the numbers $a(1 - b)$ and $b(1 - a)$ can be greater than $\frac{1}{4}$.

4. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quad-rant the triangle with the smallest area.

5. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

6. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in th tank and $H(t)$ be the height of the water in the tank at time t.
(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
(b) Is $V''(t)$ positive, negative, or zero? Explain.
(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarter full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why

7. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

8. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

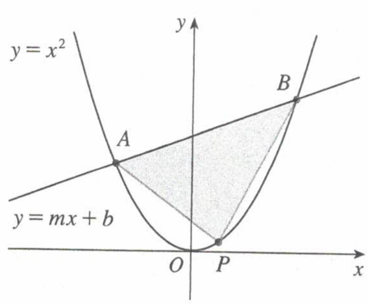


FIGURE FOR PROBLEM 9

9. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B. (See the figure.) Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

10. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty} [f(x) + x] = 0$.

11. Determine the values of the number a for which the function f has no critical number:

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

12. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

13. Let ABC be a triangle with $\angle BAC = 120°$ and $|AB| \cdot |AC| = 1$.
(a) Express the length of the angle bisector AD in terms of $x = |AB|$.
(b) Find the largest possible value of $|AD|$.

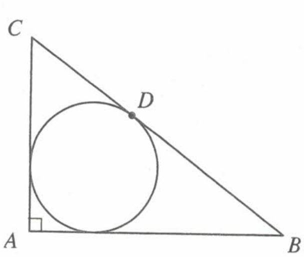


FIGURE FOR PROBLEM 14

14. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
(c) If a is fixed and θ varies, find the maximum value of r.

15. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
(a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
(b) Express da/dt in terms of the quantities in part (a).

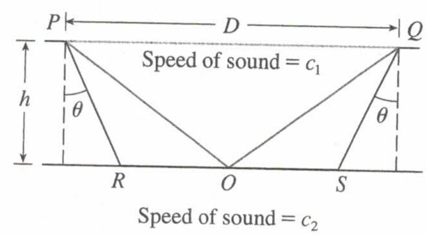
16. *ABCD* is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from *B* to *D* with center *A*. The piece of paper is folded along *EF*, with *E* on *AB* and *F* on *AD*, so that *A* falls on the quarter-circle. Determine the maximum and minimum areas that the triangle *AEF* can have.

17. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.

(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.

(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of the earth's crust, whether searching for oil or examining fault lines.

18. For what values of c is there a straight line that intersects the curve $y = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points?

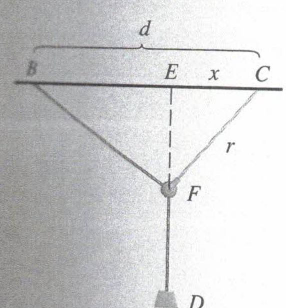


FIGURE FOR PROBLEM 19

19. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

20. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon? (A regular n-gon is a polygon with n equal sides and angles.) (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

21. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

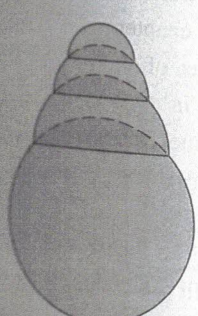


FIGURE FOR PROBLEM 22

22. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.